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Computable Bounds for Eigenvalues and Eigenfunctions of Elliptic Differential Operators

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Summary. We are concerned with bounds for the error between given approximations and the exact eigenvalues and eigenfunctions of self-adjoint operators in Hilbert spaces. The case is included where the approximations of the eigenfunctions don't belong to the domain of definition of the operator. For the eigenvalue problem with symmetric elliptic differential operators these bounds cover the case where the trial functions don't satisfy the boundary conditions of the problem. The error bounds suggest a certain defect-minimization method for solving the eigenvalue problems. The method is applied to the membrane problem.

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1. Introduction

Given a linear symmetric operator T in a Hilbert space H we are interested in bounds for eigenvalues and eigenfunctions of T by means of the defect

$$Tu - \lambda u,$$

with (λ, u) an approximate eigenpair. For the algebraic eigenvalue problem, i.e. T a symmetric $n \times n$-matrix and $H = \mathbb{R}^n$, such bounds are derived in the paper of Wilkinson [25].

These results have been generalized in various directions, see e.g. [14, 2, 22], and the references given in these papers. We are interested in results applicable to eigenvalue problems with an elliptic differential operator $T = L$. In these problems the eigenfunctions must satisfy specific boundary conditions. Most of the theorems in the literature don't cover the case, where the approximating functions u don't fit the boundary conditions or more generally, where the approximations u don't belong to the domain $D(T)$ of the symmetric operator T. In the following, we mainly direct our attention to results comprising the case $u \notin D(T)$.

In this direction in [6] bounds for eigenvalues and in [19] bounds for eigenvalues and eigenfunctions are given in the case that the functions u satisfy $Lu-\lambda u=0$. In [20, 16] bounds for eigenvalues are obtained in the general case that u satisfies neither the equation $Lu-\lambda u=0$ nor the requested boundary conditions. In Sect. 2, by a unifying treatment bounds for eigenvalues and eigenfunctions in the general case are given (cf. Theorem 2) including all these results. Moreover, bounds improved with the help of Rayleigh's quotient are derived (cf. Theorem 4). For completeness we also give the corresponding results (Theorem 1, Theorem 3) restricted to the case $u \in D(T)$.

The method depends basically on a Hilbert space approach. Therefore, in Sect. 2 all results are formulated and proved in a general Hilbert space setting.

Section 3 is concerned with the application of the theory to elliptic eigenvalue problems.

Finally, Sect. 4 is devoted to the numerical aspects of the approach.

2. Bounds for Eigenvalues and Eigenfunctions of Linear Self-Adjoint Operators in a Hilbert Space

In the following let H be a real Hilbert space and T a linear self-adjoint operator,

$$T: D(T) \to H$$

defined on a domain $D(T)$ dense in H. With the inner product $\langle \cdot, \cdot \rangle$ in H, $\|u\|$ denotes the norm

$$\|u\| = \langle u, u \rangle^{1/2}, \; u \in H.$$

For shortness we make the assumption

A: The operator T has a pure point spectrum

$$\sigma(T) = \{\tilde{\lambda}_\nu, \; \nu \in \mathbb{N}\}$$

with no finite limit point and finite multiplicities of the eigenvalues $\tilde{\lambda}_\nu$. Let $\{\lambda_\nu\}_1^\infty$ denote the sequence of the eigenvalues $\tilde{\lambda}_\nu$ counted according to their multiplicities. Moreover, we suppose that the corresponding set of orthonormalized eigenfunctions u_ν, $\nu \in \mathbb{N}$ forms a complete system in H.

Remark 1. Most of the results of this section could be proved under weaker assumptions. For operators T with continuous spectrum, for instance, the representation of Tu as a sum

$$Tu = \sum_\nu \lambda_\nu \langle u, u_\nu \rangle u_\nu$$

must be replaced in the proofs by the integral

$$Tu = \sum_{-\infty}^{\infty} \lambda \, d(E(\lambda) u)$$

where E is the spectral family corresponding to T (cf. [1, 24]). Under assumption A the self-adjointness condition on T can be replaced by the (weaker) condition that T is symmetric, i.e.

$$\langle u, Tv\rangle = \langle Tu, v\rangle \qquad \text{for all } u, v \in D(T).$$

In fact, it can be shown that for such a symmetric operator T satisfying A there always exists a self-adjoint extension $\bar{T}$ (cf. [3]).

Let us consider the eigenvalue problem

$$Tu = \lambda u.$$

An error estimate for approximate solutions $\lambda^* \in \mathbb{R}$, $v^* \in D(T)$ by means of the defect $\|Tv^* - \lambda^* v^*\|$ is given in

Theorem 1. *Let $v^* \in D(T)$, $\|v^*\| = 1$, $\lambda^* \in \mathbb{R}$ be given satisfying*

$$\|Tv^* - \lambda^* v^*\| = \varepsilon.$$

Then there exists an eigenvalue λ_k of T and a corresponding eigenfunction v_k such that

$$|\lambda_k - \lambda^*| \leqq \varepsilon \tag{2.1}$$

and

$$\|v_k - v^*\| \leqq \frac{\varepsilon}{d(\lambda^*)}\left[1 + \frac{\varepsilon^2}{d(\lambda^*)^2}\right]^{1/2} \tag{2.2}$$

holds with

$$d(\lambda^*) := \min_{\lambda_\nu \neq \lambda_k} |\lambda_\nu - \lambda^*|. \tag{2.3}$$

Proof. Using the expansions $v^* = \sum_\nu x_\nu u_\nu$, $Tv^* = \sum_\nu \lambda_\nu x_\nu u_\nu$ in terms of the eigenfunctions u_ν it follows from

$$1 = \|v^*\|^2 = \sum_\nu x_\nu^2, \qquad \langle (T - \lambda^* I) v^*, u_\nu \rangle = (\lambda_\nu - \lambda^*) x_\nu$$

that

$$\begin{aligned} \varepsilon^2 \geqq \|Tv^* - \lambda^* v^*\|^2 &= \langle (T-\lambda^* I) v^*, (T - \lambda^* I) v^* \rangle \\ &= \sum_\nu (\lambda_\nu - \lambda^*)^2 x_\nu^2 \\ &\geqq \min_{x_\nu \neq 0} (\lambda_\nu - \lambda^*)^2. \end{aligned} \tag{2.4}$$

Hence with an eigenvalue λ_k of T satisfying

$$|\lambda_k - \lambda^*| = \min_{x_\nu \neq 0} |\lambda_\nu - \lambda^*|$$

(2.1) is valid. Now let $P_k v^*$ denote the projection of v^* into the eigenspace of λ_k,

$$P_k v^* = \sum_{\lambda_\nu = \lambda_k} x_\nu u_\nu .$$

We will show that (2.2) holds with

$$v_k = \frac{P_k v^*}{\|P_k v^*\|}.$$

Since

$$\langle P_k v^*, v^* \rangle = \langle P_k v^*, P_k v^* \rangle = \sum_{\lambda_\nu = \lambda_k} x_\nu^2 = (1 - \sum_{\lambda_\nu \neq \lambda_k} x_\nu^2)$$

we obtain

$$\begin{aligned} \langle v_k - v^*, v_k - v^* \rangle &= \langle v^*, v^* \rangle + \frac{\langle P_k v^*, P_k v^* \rangle}{\|P_k v^*\|^2} - \frac{2\langle v^*, P_k v^* \rangle}{\|P_k v^*\|} \\ &= 2 - 2\|P_k v^*\| \\ &= 2[1 - (\sum_{\lambda_\nu = \lambda_k} x_\nu^2)^{1/2}] \\ &= 2[1 - (1 - \sum_{\lambda_\nu \neq \lambda_k} x_\nu^2)^{1/2}]. \end{aligned}$$

Setting $a := \sum_{\lambda_\nu \neq \lambda_k} x_\nu^2 < 1$ and using

$$(1-a)^{1/2} \geqq 1 - \tfrac{1}{2} a(1+a), \qquad \text{for } a \in \mathbb{R}, |a| < 1,$$

it follows that

$$\|v_k - v^*\|^2 = 2(1 - (1-a)^{1/2}) \leqq a(1+a). \tag{2.5}$$

Moreover, by (2.4) and (2.3) we find

$$\begin{aligned} a = \sum_{\lambda_\nu \neq \lambda_k} x_\nu^2 \leqq \sum_{\lambda_\nu \neq \lambda_k} x_\nu^2 \frac{(\lambda_\nu - \lambda^*)^2}{d(\lambda^*)^2} &= \frac{1}{d(\lambda^*)^2} \sum_{\lambda_\nu \neq \lambda_k} x_\nu^2 (\lambda_\nu - \lambda^*)^2 \\ &\leqq \frac{\varepsilon^2}{d(\lambda^*)^2}. \end{aligned}$$

Thus, by (2.5) relation (2.2) is proved. □

For more general results cf. e.g. [14, 2], where in [14] bounds for eigenvalues and in [2] bounds for eigenfunctions are given for whole invariant subspaces $S \subset D(T)$ of T and (2.1) resp. (2.2) appear as special cases.

Now we turn to the case of trial functions $u \notin D(T)$. Therefore, let in the sequel $\tilde{T}$ denote some extension of T to a domain $D(\tilde{T})$, $D(T) \subset D(\tilde{T}) \subset H$. Thereby $\tilde{T}$ is no longer assumed to be symmetric on the whole set $D(\tilde{T})$. However, in the case $u^* \in D(\tilde{T}) \setminus D(T)$ the defect $\tilde{T}u^* - \lambda^* u^*$ still leads to error bounds for eigenpairs.

Theorem 2. *Given $u^* \in D(\tilde{T}) \setminus D(T)$, $\|u^*\| = 1$ and $\lambda^* \in \mathbb{R}$, define the function r by*

$$r := \tilde{T}u^* - \lambda^* u^*.$$

a) Let $w \in D(\tilde{T})$ be a function satisfying

$$\tilde{T}w = 0, \qquad u^* - w \in D(T). \tag{2.6}$$

Then there exists an eigenpair (λ_k, v_k) of T such that

$$|\lambda_k - \lambda^*| \leqq \frac{\|r + \lambda^* w\|}{\|u^* - w\|} \tag{2.7}$$

$$\|v_k - u^*\| \leqq \varepsilon_1 (1 + \varepsilon_1^2)^{1/2} \tag{2.8}$$

where

$$\varepsilon_1 = \frac{\|r\|}{d(\lambda^*)} + \frac{\|w\|}{\tilde{d}(\lambda^*)}, \qquad \tilde{d}(\lambda^*) = \min_{\lambda_\nu \neq \lambda_k} \frac{|\lambda_\nu - \lambda^*|}{|\lambda_\nu|}$$

and $d(\lambda^*)$ is defined by (2.3).

b) Let $R \in D(\tilde{T})$ be a solution of

$$\tilde{T}R = \tilde{T}u^* - \lambda^* u^*, \qquad u^* - R \in D(T). \tag{2.9}$$

Then there exists an eigenpair (λ_k, v_k) such that

$$\left| \frac{\lambda_k - \lambda^*}{\lambda_k} \right| \leqq \|R\|, \tag{2.10}$$

$$\|v_k - u^*\| \leqq \varepsilon_2 (1 + \varepsilon_2^2)^{1/2} \tag{2.11}$$

where

$$\varepsilon_2 = \frac{\|R\|}{\tilde{d}(\lambda^*)}.$$

Proof. Setting $v^* = \dfrac{u^* - w}{\|u^* - w\|} \in D(T)$ we have $\|v^*\| = 1$ and

$$Tv^* - \lambda^* v^* = \frac{r + \lambda^* w}{\|u^* - w\|},$$

which by Theorem 1 yields (2.7). Making use of the expansions

$$r = \sum_\nu r_\nu u_\nu, \quad R = \sum_\nu R_\nu u_\nu, \quad w = \sum_\nu w_\nu u_\nu, \quad u^* = \sum_\nu x_\nu u_\nu$$

we find

$$\begin{aligned} \lambda_\nu (x_\nu - w_\nu) &= \langle u^* - w, Tu_\nu \rangle = \langle T(u^* - w), u_\nu \rangle \\ &= \langle \lambda^* u^* + r, u_\nu \rangle = \lambda^* x_\nu + r_\nu \end{aligned}$$

or

$$(\lambda_\nu - \lambda^*)\, x_\nu = \lambda_\nu w_\nu + r_\nu .$$

By applying the Cauchy-Schwarz inequality we get

$$a := \sum_{\lambda_\nu \neq \lambda_k} x_\nu^2 = \sum_{\lambda_\nu \neq \lambda_k} \left[\frac{\lambda_\nu}{\lambda_\nu - \lambda^*}\, w_\nu + \frac{r_\nu}{\lambda_\nu - \lambda^*}\right]^2$$

$$\leqq \left[\frac{[\sum_\nu w_\nu^2]^{1/2}}{\tilde{d}(\lambda^*)} + \frac{[\sum_\nu r_\nu^2]^{1/2}}{d(\lambda^*)}\right]^2 .$$

Now just as in the proof of Theorem 1 by setting $v_k := P_k u^* / \|P_k u^*\|$ we obtain inequality (2.5),

$$\|v_k - u^*\|^2 \leqq a(1+a)$$

which proves (2.8). Similarly with

$$\lambda_\nu (x_\nu - R_\nu) = \langle u^* - R,\, T u_\nu \rangle = \langle T(u^* - R),\, u_\nu \rangle = \lambda^* x_\nu ,$$

or

$$(\lambda_\nu - \lambda^*)\, x_\nu = \lambda_\nu R_\nu , \tag{2.12}$$

we deduce

$$a := \sum_{\lambda_\nu \neq \lambda_k} \left[\frac{\lambda_\nu}{\lambda_\nu - \lambda^*}\right]^2 R_\nu^2 \leqq \left[\frac{\|R\|}{\tilde{d}(\lambda^*)}\right]^2$$

and then (2.11). Since under assumption A $\lim_{\nu \to \infty} |\lambda_\nu| = \infty$, there exists a $k \in \mathbb{N}$ such that

$$\frac{|\lambda_k - \lambda^*|}{|\lambda_k|} = \min_{\lambda_\nu \neq 0} \frac{|\lambda_\nu - \lambda^*|}{|\lambda_\nu|} .$$

Hence by (2.12) it follows that

$$\left|\frac{\lambda_k - \lambda^*}{\lambda_k}\right| |x_\nu| \leqq |R_\nu| \quad \text{for all } \nu \in \mathbb{N}.$$

(Here we tacitly assume $\lambda_\nu^* \neq 0$ for all $\nu \in \mathbb{N}$.) Consequently

$$\left|\frac{\lambda_k - \lambda^*}{\lambda_k}\right|^2 = \left|\frac{\lambda_k - \lambda^*}{\lambda_k}\right|^2 \sum_\nu x_\nu^2 \leqq \sum_\nu R_\nu^2 = \|R\|^2$$

establishing (2.10). □

The paper [8] contains a theorem (Satz 19) in a more general context including (2.10).

For functions $v \in D(T)$ let $\rho(v)$ denote the Rayleigh quotient of v with respect to the operator T,

$$\rho(v) = \frac{\langle Tv, v\rangle}{\langle v, v\rangle}. \tag{2.13}$$

In many cases sharper error bounds for eigenvalues than (2.1) can be obtained by means of the defect $\|Tv^* - \rho^* v^*\|$ if ρ^* is the Rayleigh quotient of v^*.

Theorem 3. *Let the inequality*

$$\|Tv^* - \rho^* v^*\| \leqq \varepsilon$$

hold with $v^* \in D(T)$, $\|v^*\| = 1$ and the Rayleigh quotient $\rho^* = \rho(v^*)$. Then there exists an eigenvalue λ_k of T satisfying

$$-\frac{\varepsilon^2}{d_+(\rho^*)} \leqq \lambda_k - \rho^* \leqq \frac{\varepsilon^2}{d_-(\rho^*)} \tag{2.14}$$

where

$$d_+(\rho^*) = \min_{\lambda_\nu > \lambda_k} |\lambda_\nu - \rho^*|, \qquad d_-(\rho^*) = \min_{\lambda_\nu < \lambda_k} |\lambda_\nu - \rho^*|. \tag{2.15}$$

Proof. The proof of the theorem can be found in [1], where this result is attributed to Kato and Temple.

In cases where $d_+(\rho^*), d_-(\rho^*) > \varepsilon$, Theorem 3 gives better bounds than Theorem 1.

Remark 2. Theorem 3 applies when solving the eigenvalue problem by the Rayleigh-Ritz method. Here approximate eigenpairs (λ^*, v*) are given as solutions $\lambda^* \in \mathbb{R}$, $v^* \in V$ of

$$\langle Tv^*, u\rangle = \lambda^* \langle v^*, u\rangle \quad \text{for all } u \in V,$$

where $V \subset D(T)$ is a space of trial functions. Thus, for such solutions (λ^*, v^*) we have

$$\lambda^* = \frac{\langle Tv^*, v^*\rangle}{\langle v^*, v^*\rangle} = \rho(v^*),$$

and the defect $\|Tv^* - \lambda^* v^*\|$ directly leads to an error bound according to (2.14).

When taking trial functions $u^* \in D(\tilde{T}) \setminus D(T)$, Theorem 3 must be modified.

Theorem 4. *Given $u^* \in D(\tilde{T}) \setminus D(T)$, $\|u^*\| = 1$, $\lambda^* \in \mathbb{R}$, define the function r by*

$$r = \tilde{T}u^* - \lambda^* u^*.$$

a) For a solution $w \in D(\tilde{T})$ of

$$\tilde{T}w = 0, \qquad u^* - w \in D(T) \tag{2.6}$$

let

$$v_1 := u^* - w, \qquad \varepsilon_1 := \frac{\|\lambda^* w + r\|}{\|u^* - w\|}. \tag{2.16}$$

Then with the Rayleigh quotient $\rho_1 = \rho(v_1)$ *given by*

$$\rho_1 = \lambda^* + \frac{\langle \lambda^* w + r, v_1 \rangle}{\langle v_1, v_1 \rangle}$$

the inequality

$$-\frac{(2\varepsilon_1)^2}{d_+(\rho_1)} \leqq \lambda_k - \rho_1 \leqq \frac{(2\varepsilon_1)^2}{d_-(\rho_1)} \tag{2.17}$$

holds for some eigenvalue λ_k *of* T.

b) For a solution $R \in D(\tilde{T})$ *of*

$$\tilde{T}R = \tilde{T}u^* - \lambda^* u^*, \qquad u^* - R \in D(T) \tag{2.9}$$

let

$$v_2 := u^* - R, \qquad \varepsilon_2 := |\lambda^*| \frac{\|R\|}{\|u^* - R\|}. \tag{2.18}$$

Then with $\rho_2 = \rho(v_2)$ *given by*

$$\rho_2 = \lambda^* + \frac{\langle \lambda^* R, v_2 \rangle}{\langle v_2, v_2 \rangle},$$

the inequality

$$-\frac{(2\varepsilon_2)^2}{d_+(\rho_2)} \leqq \lambda_k - \rho_2 \leqq \frac{(2\varepsilon_2)^2}{d_-(\rho_2)} \tag{2.19}$$

is valid.

Proof. With

$$\rho_1 = \rho(v_1) = \frac{\langle Tv_1, v_1 \rangle}{\langle v_1, v_1 \rangle} = \frac{\langle \lambda^*(v_1 + w) + r, v_1 \rangle}{\langle v_1, v_1 \rangle}$$
$$= \lambda^* + \frac{\langle \lambda^* w + r, v_1 \rangle}{\langle v_1, v_1 \rangle},$$

by the Cauchy-Schwarz inequality we find

$$\|Tv_1 - \rho_1 v_1\| = \left\| Tv_1 - \lambda^* v_1 - \frac{\langle \lambda^* w + r, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 \right\|$$
$$\leqq \|Tv_1 - \lambda^* v_1\| + \left| \frac{\langle \lambda^* w + r, v_1 \rangle}{\langle v_1, v_1 \rangle} \right| \|v_1\|$$
$$\leqq 2\|\lambda^* w + r\|.$$

Now by normalizing $\tilde{v}_1 = \frac{v_1}{\|v_1\|}$, we obtain $\|T\tilde{v}_1 - \rho_1 \tilde{v}_1\| \leqq 2\varepsilon_1$, and thus (2.17) follows from Theorem 3. In the same way, (2.19) is proved by observing that

$$\rho_2 = \rho(v_2) = \frac{\langle Tv_2, v_2\rangle}{\langle v_2, v_2\rangle} = \frac{\langle \lambda^*(v_2+R), v_2\rangle}{\langle v_2, v_2\rangle} = \lambda^* \left[1 + \frac{\langle R, v_2\rangle}{\langle v_2, v_2\rangle}\right]$$

and

$$\begin{aligned}\|Tv_2 - \rho_2 v_2\| &= \left\| Tv_2 - \lambda^* v_2 - \lambda^* \frac{\langle R, v_2\rangle}{\langle v_2, v_2\rangle} v_2 \right\| \\ &\leqq \|Tv_2 - \lambda^* v_2\| + |\lambda^*|\,\|R\| \leqq 2|\lambda^*|\,\|R\|. \quad \square\end{aligned}$$

We emphasize that to apply Theorem 2 we only need an upper bound for the norms $\|w\|$, $\|R\|$ of the solutions of (2.6) and (2.9), whereas in Theorem 4 the functions w, R are needed explicitly for the computation of the Rayleigh quotients ρ_1, ρ_2. Clearly in practice the exact solutions w, R are replaced by approximations $\tilde{w}$, $\tilde{R}$. Accordingly v_1, ρ_1 and v_2, ρ_2 are replaced by $\tilde{v}_1 := u^* - \tilde{w}$,

$$\tilde{\rho}_1 = \lambda^* + \frac{\langle \lambda^* \tilde{w} + r, \tilde{v}_1\rangle}{\langle \tilde{v}_1, \tilde{v}_1\rangle} \tag{2.20}$$

and $\tilde{v}_2 := u^* - \tilde{R}$,

$$\tilde{\rho}_2 = \lambda^* + \lambda^* \frac{\langle \tilde{R}, \tilde{v}_2\rangle}{\langle \tilde{v}_2, \tilde{v}_2\rangle}. \tag{2.21}$$

To maintain explicit bounds in this case it is clear by observing

$$\lambda_k - \rho_i - |\rho_i - \tilde{\rho}_i| \leqq \lambda_k - \tilde{\rho}_i \leqq \lambda_k - \rho_i + |\rho_i - \tilde{\rho}_i| \qquad i = 1, 2 \tag{2.22}$$

that we need upper bounds for $|\rho_i - \tilde{\rho}_i|$, i = 1, 2, in terms of computable quantities.

Lemma 1. *Let $\tilde{w}$ and $\tilde{R}$ be approximate solutions of* (2.6) *and* (2.9), *respectively. Then with $d_1 := w - \tilde{w}$ resp. $d_2 := R - \tilde{R}$ and $\tilde{\rho}_1, \tilde{\rho}_2$ given by* (2.20), (2.21) *we have*

$$|\rho_1 - \tilde{\rho}_1| \leqq \|d_1\| \frac{|\lambda^*| \left[\|\tilde{v}_1\| + 3\|\tilde{w}\| + \|d_1\| \left[1 + \frac{\|\tilde{w}\|}{\|\tilde{v}_1\|}\right]\right] + \|r\| \left[3 + \frac{\|d_1\|}{\|\tilde{v}_1\|}\right]}{\|\tilde{v}_1 - d_1\|^2} \tag{2.23}$$

and

$$|\rho_2 - \tilde{\rho}_2| \leqq \|d_2\|\,|\lambda^*| \frac{\|\tilde{v}_2\| + 3\|\tilde{R}\| + \|d_2\| \left[1 + \frac{\|\tilde{R}\|}{\|\tilde{v}_2\|}\right]}{\|\tilde{v}_2 - d_2\|^2} \tag{2.24}$$

Proof. Since $v_1 = u^* - w = \tilde{v}_1 - d_1$ we find by a short calculation

$$\begin{aligned}|\rho_1 - \tilde{\rho}_1| &= \left| \frac{\langle \lambda^* w + r, v_1\rangle}{\langle v_1, v_1\rangle} - \frac{\langle \lambda^* \tilde{w} + r, \tilde{v}_1\rangle}{\langle \tilde{v}_1, \tilde{v}_1\rangle} \right| \\ &= \left| \frac{\langle \lambda^* \tilde{w} + \lambda^* d_1 + r, \tilde{v}_1 - d_1\rangle}{\langle \tilde{v}_1 - d_1, \tilde{v}_1 - d_1\rangle} - \frac{\langle \lambda^* \tilde{w} + r, \tilde{v}_1\rangle}{\langle \tilde{v}_1, \tilde{v}_1\rangle} \right| \\ &\leqq \|d_1\| \frac{|\lambda^*| \left[\|\tilde{v}_1\| + 3\|\tilde{w}\| + \|d_1\| \left[1 + \frac{\|\tilde{w}\|}{\|\tilde{v}_1\|}\right]\right] + \|r\| \left[3 + \frac{\|d_1\|}{\|\tilde{v}_1\|}\right]}{\|\tilde{v}_1 - d_1\|^2},\end{aligned}$$

and (2.24) is obtained by similar manipulations. □

For approximations (λ^*, u^*) close to an eigenpair (λ_k, v_k) the quantities $\|w\|$, $\|R\|$ become nearly zero and $\|\tilde{v}_i\|$ nearly 1 provided that $\|d_i\|$ is small. Thus, the difference $|\rho_i - \tilde{\rho}_i|$ essentially reads

$$|\rho_i - \tilde{\rho}_i| \leqq \|d_i\| \, |\lambda^*| \qquad i = 1, 2.$$

Remark 3. Let an approximate eigenpair (λ^*, v^*) and the expansion $v^* = \sum_\nu x_\nu u_\nu$ in terms of eigenfunctions be given. Then in Theorem 1, instead by (2.3) $d(\lambda^*)$ could be defined by

$$d(\lambda^*) = \min_{\substack{\lambda_\nu \neq \lambda_k \\ x_\nu \neq 0}} |\lambda_\nu - \lambda^*| \tag{2.25}$$

(cf. the proof of Theorem 1). This modification is especially useful when considering eigenvalue problems with differential operators on domains having certain symmetries. Then often the set of eigenfunctions $\{u_\nu, \nu \in \mathbb{N}\}$ can be subdivided into subsets of functions with corresponding symmetries (cf. [17, 10]) and it is possible to choose trial functions v^* with the symmetry of such a subset S. Consequently v^* allows a representation

$$v^* = \sum_{u_\nu \in S} x_\nu u_\nu,$$

and (2.25) possibly leads to sharper bounds for eigenvectors in (2.2). The same modification applies to the definition of $\tilde{d}(\lambda^*)$ in Theorem 2a) and $d_+(\rho^*)$, $d_-(\rho^*)$ in (2.15).

3. Bounds for Eigenvalues and Eigenfunctions of Elliptic Differential Operators

This section is concerned with the application of the previous results to linear elliptic differential operators. We are especially interested in differential operators L with analytic coefficients on regions G with piecewise analytic boundary ∂G. Then singular behavior (in the derivatives) of the eigenfunctions of L can only occur at the corners of ∂G.

Let L denote a linear symmetric differential operator,

$$Lu = -\sum_{\nu, \mu = 1}^{2} (a_{\nu\mu} u_{x_\nu})_{x_\mu} + cu \tag{3.1}$$

with functions c, $a_{\nu\mu}$ analytic on an open set $D \subset \mathbb{R}^2$, $\bar{G} \subset D$ satisfying $c(x) \geqq 0$, $a_{12}(x) = a_{21}(x)$ for $x = (x_1, x_2) \in \bar{G}$ and the strict ellipticity condition

$$\sum_{\nu, \mu = 1}^{2} a_{\nu\mu}(x) \xi_\nu \xi_\mu \geqq \gamma(\xi_1^2 + \xi_2^2) \qquad \text{for all } x \in G,\ \xi \in \mathbb{R}^2 \tag{3.2}$$

with $\gamma>0$. Thereby we assume that G is a bounded connected open set with piecewise analytic boundary ∂G, i.e. there exists a number k of analytic curves $\Gamma_1, \ldots, \Gamma_k$, such that every $x\in\partial G$ is an inner point of one of these curves. At the points $S_1, \ldots, S_k$ of intersection the tangents to the meeting arcs are supposed to form corners with interior angles ω_ν, $0<\omega_\nu<2\pi$, $\nu=1(1)k$. (As usual a function is said to be analytic if locally it is given by a convergent power series, and by an analytic curve we mean a curve parametrizable by analytic functions.)

We consider the eigenvalue problem of finding solutions $\lambda\in\mathbb{R}$, $u\neq 0$, of

$$\begin{aligned} Lu(x)-\lambda u(x)&=0 \quad \text{for } x\in G\\ u(x)&=0 \quad \text{for } x\in\partial G. \end{aligned} \tag{3.3}$$

We take $H=L_2(G)$, the Hilbert space of all square integrable functions with inner product

$$\langle u, v\rangle = \int_G uv\,dx \Big/ \int_G dx$$

Given $p>1$ we denote by $W_p^1(G)$ the Sobolev space of functions u for which the derivatives $\frac{\partial}{\partial x_1}u(x)$, $\frac{\partial}{\partial x_2}u(x)$ (in the sense of distributions) are elements of $L_p(G)$ and by $\mathring{W}_p^1(G)$ the closure in $W_p^1(G)$ of the set of all C^∞ functions with compact support in G. Furthermore, as the domain of L we define

$$D_p(L):=\{u\in\mathring{W}_p^1(G)\colon Lu\in L_2(G)\}.$$

Then (cf. [9, Theorem 1.5.3.11]) for any $p>2$ the symmetry

$$\langle Lu, v\rangle = \langle u, Lv\rangle \qquad \text{for all } u, v\in D_p(L) \tag{3.4}$$

holds. Finally we denote by $\tilde{L}$ the extension of L to the domain

$$D_p(\tilde{L}):=\{u\in W_p^1(G)\colon Lu\in L_2(G)\}.$$

In the following remark we explain why we did not content ourselves with a domain $D=\mathring{W}_2^2(G)$ (resp. $\tilde{D}=W_2^2(G)$) of definition of L (resp. $\tilde{L}$). For the space D the proof of symmetry is much simpler than for the larger spaces $D_p(L)$ (cf. [9, Lemma 1.5.3.2]).

Remark 4. From [18] we know that the solutions u of equations $Lu-\lambda u=f$ on regions G with piecewise analytic boundary and f analytic on $\bar{G}$ are analytic throughout $\bar{G}$ except at the corners of ∂G. At a corner $s=(s_1, s_2)$ of ∂G with interior angle ω, $0<\omega<2\pi$ the function u allows, setting (in the case $L=-\Delta$) $\alpha:=\frac{\pi}{\omega}$, for any $k\in\mathbb{R}$ a development (in (r, φ) polar-coordinates, $x_1=s_1+r\cos\varphi$, $x_2=s_2+r\sin\varphi$; $\varphi=0$, ω corresponding to the tangents to ∂G at S)

$$u(r,\varphi)=\sum_{\ell\alpha+n<k} r^{\ell\alpha+n}(a_{\ell n}\sin((\ell\alpha+n)\varphi)+b_{\ell n}\cos((\ell\alpha+n)\varphi))+0(r^k)$$

in the case where α is irrational and

$$u(r, \varphi) = \sum_{\ell\alpha + nq < k} r^{\ell\alpha + nq} \log^q r\, \psi_{\ell n}(\varphi) + 0(r^k)$$

with analytic functions $\psi_{\ell n}$ if $\alpha = q/m$ is rational (cf. also [4, 23, 9, Chapters 4, 5]). For modifications when $L \neq -\Delta$, we refer to [9, p. 265ff]. Moreover, by the theory of [9, Chapter 5] for the eigenvalue problem $-\Delta u = \lambda u$ with boundary condition $u_{|\partial G} = 0$, the development of u at corner s has the first form with $b_{\ell n} = 0$ for all α, e.g. the functions $\log^q r$ don't occur. In any case, typically such an expansion begins with a term

$$u^0(r, \varphi) = r^\alpha (a \sin \alpha\varphi + b \cos \alpha\varphi).$$

Now (cf. [9, p. 35]) if $\alpha \geqq 1$, i.e. $\omega \leqq \pi$, then u^0 (and so u) is in $W_2^2(G)$. But for $\alpha < 1$, i.e. $\omega > \pi$, $u^0 \notin W_2^2(G)$. Fortunately $u^0 \in W_p^1(G)$ if $\alpha > 1 - \dfrac{2}{p}$. Since $\omega < 2\pi$, i.e. $\alpha > 1/2$, $u^0 \in W_p^1(G)$ in any case if $p \leqq 4$. Therefore, since we want to use trial functions with the same type of singularity at the corners of ∂G as the eigenfunctions without losing symmetry (3.4), we can choose as domain of L the space $D_p(L)$, for a suitable p $2 < p$.

In the sequel, according to the preceding remark we take the domains of definition of L and $\tilde{L}$ to be

$$D(L) = D_p(L), \qquad D(\tilde{L}) = D_p(\tilde{L})$$

for some fixed p, $2 < p$. Notice that $W_p^1(G) \subset C(\bar{G})$ for $p > 2$ (cf. [9, Theorem 1.4.5.2]). Thus, the functions $u \in D(L)$ satisfy the boundary condition $u_{|\partial G} = 0$ in the classical sense. It can be shown that in our situation assumption A of Sect. 2 holds for the operator L in (3.1). This can be achieved by showing first that the eigenpairs (λ_ν, u_ν), $u_\nu \in \mathring{W}_2^1(G)$ of the variational eigenvalue problem corresponding to (3.3) satisfy assumption A. Then by the results of [9] it follows that $u_\nu \in D_p(L)$ for suitable $p > 2$. In [5] based on the study of the inverse operator of Δ (i.e. Green's function) it is shown, that the space $D_0 := \{u \in C^2(G) \cap C(\bar{G});\ u_{|\partial G} = 0\}$ could be taken as domain of definition of Δ as well. Notice, that neither $D(L) \subset D_0$ nor clearly the converse.

Consequently all results of Sect. 2 apply to the eigenvalue problem (3.3).

Note that the problems (2.6), (2.9) in Theorems 2 and 4 are in this case the following boundary value problems:

For given $\lambda^* \in \mathbb{R}$ and $u^* \in D(\tilde{L})$ w, $R \in D(\tilde{L})$ are solutions of

$$\tilde{L}w = 0 \quad \text{on } G, \qquad u^* = w \quad \text{on } \partial G, \tag{3.5}$$

$$\tilde{L}R = \tilde{L}u^* - \lambda^* u^* \quad \text{on } G, \quad u^* = R \quad \text{on } \partial G. \tag{3.6}$$

Again, the existence and uniqueness of solutions w, R is established in [9].

As mentioned above, the bounds of Theorem 2 only require upper bounds on $\|w\|$, $\|R\|$. We therefore briefly deal with such bounds.

By the Maximum principle for solutions of (3.5) (cf. [7]) the relation

$$\|w\| \leqq \max_{x \in \bar{G}} |w(x)| \leqq \max_{x \in \partial G} |w(x)| = \max_{x \in \partial G} |u^*(x)| \tag{3.7}$$

is valid. Another upper bound for $\|w\|$ is given by the inequality

$$\int_G w^2\, dx \leqq c \int_{\partial G} w^2\, ds \quad \text{for all } w, \text{ satisfying } \tilde{L}w = 0 \quad \text{on } G,$$

with a constant c depending only on L and G, leading to the bound

$$\|w\| \leqq \sqrt{c}\Big(\int_{\partial G} (u^*)^2\, ds\Big)^{1/2} \tag{3.8}$$

for the solutions w of (3.5).

An optimal constant c could be found by solving the problem

$$\max \int_G w^2\, dx \quad \text{subject to} \int_{\partial G} w^2\, ds = 1, \quad \tilde{L}w = 0 \quad \text{on } G.$$

For upper bounds for minimal constants c we refer to [15]. If $L = -\Delta$ and G is convex, for instance, then

$$c \leqq \sqrt{\frac{A}{2\pi}}, \qquad A \text{ the area of } G.$$

We notice that for eigenpairs (λ, u) of (3.3) by the Divergence Theorem and (3.2) we get

$$\langle Lu, u\rangle = \int_G \sum_{\nu,\mu} a_{\nu\mu} u_{x_\nu} u_{x_\mu}\, dx + \int_G c u^2\, dx > 0$$

and consequently

$$\lambda = \frac{\langle Lu, u\rangle}{\langle u, u\rangle} > 0. \tag{3.9}$$

Let λ_1 denote the smallest eigenvalue of L. By (3.9) clearly $\lambda_1 > 0$.

In the following lemma, by the same method as in the previous section, we will obtain an error bound for solutions of boundary value problems.

Lemma 2. *Let functions $f \in C(\bar{G})$, $g \in C(\partial G)$ and the boundary value problem*

$$\begin{aligned} \tilde{L}u(x) &= f(x), \quad x \in G \\ u(x) &= g(x), \quad x \in \partial G \end{aligned} \tag{3.10}$$

be given. Let $R^* \in D(\tilde{L})$ *be an approximation to the solution* u *of* (3.10). *Then with the solution* w *of*

$$\tilde{L}w = 0 \quad on\ G, \qquad w = R^* - g \quad on\ \partial G$$

we have

$$\|u - R^*\| \leqq \frac{\|\tilde{L}R^* - f\|}{\lambda_1} + \|w\|. \tag{3.11}$$

Proof. Setting $r := \tilde{L}R^* - f$ since $R^* - u - w \in D(L)$ we find

$$\lambda_\nu \langle R^* - u - w, u_\nu \rangle = \langle R^* - u - w, Lu_\nu \rangle = \langle \tilde{L}R^* - \tilde{L}u, u_\nu \rangle = \langle r, u_\nu \rangle$$

or

$$\langle R^* - u, u_\nu \rangle = \frac{\langle r, u_\nu \rangle}{\lambda_\nu} + \langle w, u_\nu \rangle \qquad \nu \in \mathbb{N}.$$

Hence by the Parceval identity and Cauchy-Schwarz inequality it follows that

$$\|u - R^*\|^2 = \sum_\nu \langle R^* - u, u_\nu \rangle^2 = \sum_\nu \left[\frac{\langle r, u_\nu \rangle}{\lambda_\nu} + \langle w, u_\nu \rangle\right]^2$$

$$\leqq \left[\left(\sum_\nu \left[\frac{\langle r, u_\nu \rangle}{\lambda_\nu}\right]^2\right)^{1/2} + \left(\sum_\nu \langle w, u_\nu \rangle^2\right)^{1/2}\right]^2$$

and

$$\|u - R^*\| \leqq \frac{1}{\lambda_1} \|\tilde{L}R^* - f\| + \|w\|. \quad \square$$

Now we apply Lemma 2 to the problem (3.10) with $f \equiv 0$ and $g \equiv 0$. Then, with the solution $u \equiv 0$ and the solution R of (3.6) we obtain inequality

$$\|R\| \leqq \frac{\|\tilde{L}R\|}{\lambda_1} + \|w\| = \frac{\|\tilde{L}u^* - \lambda^* u^*\|}{\lambda_1} + \|w\|. \tag{3.12}$$

Here in addition the upper bound (3.7) or (3.8) for $\|w\|$ could be considered.

Summarizing we obtain the following bounds in terms of approximate eigenpairs (λ^*, u^*), $\|u^*\| = 1$.

Setting

$$\begin{aligned} &\|u\|_\infty := \max_{x \in \bar{G}} |u(x)|, \\ &\|u\|_{\partial G} := \Big(\int_{\partial G} u(s)^2\, ds\Big)^{1/2}, \qquad \|u\|_{\partial G, \infty} := \max_{x \in \partial G} |u(x)|, \end{aligned} \tag{3.13}$$

(2.7) together with (3.7) resp. (3.8) yield

$$|\lambda_k - \lambda^*| \leqq \frac{\|\tilde{L}u^* - \lambda^* u^*\|_\infty + \lambda^* \|u^*\|_{\partial G,\infty}}{1 - \|u^*\|_{\partial G,\infty}}, \tag{3.14}$$

resp.

$$|\lambda_k - \lambda^*| \leqq \frac{\|\tilde{L}u^* - \lambda^* u^*\| + \lambda^* \sqrt{c}\, \|u\|_{\partial G}}{1 - \sqrt{c}\, \|u\|_{\partial G}}, \tag{3.15}$$

provided that $\|u^*\|_{\partial G,\infty} < 1$, resp. $\sqrt{c}\, \|u^*\|_{\partial G} < 1$.

Using (3.12) and (3.8) the relative bound (2.10) reads

$$\left|\frac{\lambda_k - \lambda^*}{\lambda_k}\right| \leqq \frac{\|\tilde{L}u^* - \lambda^* u\|}{\lambda_1} + \sqrt{c}\, \|u^*\|_{\partial G}.$$

On the other hand from (2.10) we find

$$|\lambda_k - \lambda^*| \leqq \frac{\lambda^* \|R\|}{1 - \|R\|},$$

which by (3.12) and (3.8) leads to the bound

$$|\lambda_k - \lambda^*| \leqq \frac{\lambda^*/\lambda_1 \|\tilde{L}u^* - \lambda^* u^*\| + \sqrt{c}\, \lambda^* \|u^*\|_{\partial G}}{1 - \|R\|} \tag{3.16}$$

provided that $\|R\| < 1$. Thus for $\lambda^* > \lambda_1$ (3.15) represents a sharper bound than (3.16).

4. Numerical Aspects

The two bounds (3.14) and (3.15) suggest two different numerical methods to solve the eigenvalue problem (3.3).

Let us choose an appropriate n-dimensional space V_n,

$$V_n = \{u_n(\alpha, x) = \sum_{\nu=1}^{n} \alpha_\nu \varphi_\nu(x) : \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n,\ x \in \bar{G}\}, \tag{4.1}$$

with appropriate trial functions $\varphi_1, \ldots, \varphi_n$. To make use of (3.14) we consider for fixed $\lambda \in \mathbb{R}$, $n \in \mathbb{N}$ the optimization problem

$P_n(\lambda)$: $\underset{(\alpha,\mu)}{\text{Min}}\ l(\alpha, \mu) = \mu$, $\quad (\alpha, \mu) \in \mathbb{R}^{n+1}$, $\quad$ subject to the constraints

$$|(\tilde{L}-\lambda I)\,u_n(\alpha, x)| = |(\tilde{L}-\lambda I)\sum_{\nu=1}^{n} \alpha_\nu \varphi_\nu(x)| \leqq \mu\rho \qquad \text{for all } x \in G \tag{4.2}$$

$$|u_n(\alpha, x)| = |\sum_{\nu=1}^{n} \alpha_\nu \varphi_\nu(x)| \leqq \mu \qquad \text{for all } x \in \partial G$$

$$\alpha_1 \geqq 1.$$

Here ρ is a weight factor, e.g. $\rho = \lambda$ (cf. (3.14)). The constraint $\alpha_1 \geqq 1$ is one of many other possible linear constraints added to exclude the zero solution.

$P_n(\lambda)$ represents a parametric linear semi-infinite programming problem (cf. [11]).

With a solution (α, μ) of $P_n(\lambda)$ (λ fixed) by normalizing

$$u_n^*(\alpha, x) := \frac{u_n(\alpha, x)}{\|u_n(a, \bullet)\|}, \qquad \mu^*(\lambda) := \frac{\mu}{\|u_n(\alpha, \bullet)\|}$$

(3.14) yields the bound

$$|\lambda_k - \lambda| \leqq \frac{\rho\mu^*(\lambda) + \lambda\mu^*(\lambda)}{1 - \mu^*(\lambda)} := \varepsilon_n(\lambda) \tag{4.3}$$

for an eigenvalue λ_k (and a bound for an eigenfunction v_k corresponding to (2.8)) provided that $\mu^*(\lambda) < 1$. Thus, to find small bounds for an eigenvalue we are interested in local minima of the right-hand side $\varepsilon_n(\lambda)$.

This optimization method has been regarded from the theoretical and practical points of view in [11,12], and has been applied to $L = -\Delta$, Δ the Laplace operator

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2},$$

using trial functions $\varphi = \varphi(\lambda, x)$ satisfying

$$\tilde{\Delta}\varphi(\lambda, x) + \lambda\varphi(\lambda, x) = 0 \qquad \text{for all } \lambda \in \mathbb{R}, \lambda > 0, x \in G.$$

In this case the constraints for $x \in G$ in (4.2) are dropped. Trial functions φ satisfying $\tilde{\Delta}\varphi(\lambda, x) + \lambda\varphi(\lambda, x) = 0$ for $\lambda > 0$ are given for example by

$$J_\alpha(\sqrt{\lambda}\, r) \begin{matrix} \cos\alpha\varphi \\ \cdot \\ \sin\alpha\varphi \end{matrix} \quad \alpha \in \mathbb{R}, \qquad (r, \varphi) \text{ polar coordinates}$$

or

$$\begin{matrix} \sin\alpha x_1 & \sin\beta x_2 \\ \sin\alpha x_1 & \cos\beta x_2 \\ \cos\alpha x_1 & \sin\beta x_2 \\ \cos\alpha x_1 & \cos\beta x_2 \end{matrix} \qquad \alpha, \beta \quad \text{such that } \alpha^2 + \beta^2 = \lambda.$$

A method based on (3.16) has been proposed in [16]. Since according to our previous remark the bound (3.15) generally is sharper than (3.16), we briefly outline this method applied to (3.15).

We want to minimize the square of the numerator of the right-hand side of (3.15).

To this end, let us consider for $\lambda \in \mathbb{R}$, $n \in \mathbb{N}$ fixed, the problem

$$Q_n(\lambda):\ \underset{\alpha}{\text{Min}}\ B(\lambda,\alpha) := 2\int_G (\tilde{L}u_n(\alpha,x) - \lambda u_n(\alpha,x))^2\,dx + 2\lambda^2 c \int_{\partial G} u_n^2(\alpha,s)\,ds$$

$$\text{subject to the constraints} \tag{4.4}$$

$$A(\alpha) = \int_G u_n^2(\alpha,x)\,dx = 1$$

with $u_n(\alpha, x) \in V_n$ (cf. (4.1)). Since

$$B(\lambda,\alpha) = \sum_{\nu,\mu=1}^{n} \alpha_\nu \alpha_\mu c_{\nu\mu}(\lambda)$$

with

$$c_{\nu\mu}(\lambda) = 2\int_G \tilde{L}\varphi_\nu \tilde{L}\varphi_\mu - \lambda(\tilde{L}\varphi_\nu\,\varphi_\mu + \varphi_\nu\,\tilde{L}\varphi_\mu) + \lambda^2 \varphi_\nu\,\varphi_\mu\,dx + \lambda^2 c \int_{\partial G} \varphi_\nu\,\varphi_\mu\,ds$$

and

$$A(\alpha) = \sum_{\nu,\mu=1}^{n} \alpha_\nu \alpha_\mu d_{\nu\mu} \qquad \text{where } d_{\nu\mu} = \int_G \varphi_\nu\,\varphi_\mu\,dx,$$

$Q(\lambda)$ represents the finite optimization problem

$$\underset{\alpha}{\text{Min}}\ \alpha^T C(\lambda)\,\alpha \qquad \text{subject to } \alpha^T A\,\alpha = 1$$

where

$$C(\lambda) = ((c_{\nu\mu}(\lambda)))_{\nu,\mu=1(1)n}, \qquad D = ((d_{\nu\mu}))_{\nu,\mu=1(1)n}.$$

with C positive semi-definite, D positive definite. By the Lagrange Theorem the value of $Q(\lambda)$ (λ fixed) is given by the smallest eigenvalue $\rho = \rho(\lambda)$ of the algebraic eigenvalue problem

$$C(\lambda)\,\alpha = \rho D\alpha. \tag{4.5}$$

By setting

$$\varepsilon_n(\lambda) := \sqrt{\rho(\lambda)} \tag{4.6}$$

and with the solution $\bar{\alpha}$ of $Q(\lambda)$

$$\bar{\varepsilon}_n(\lambda) := \sqrt{c}\,\Big(\int_{\partial G} u_n^2(\bar{\alpha},s)\,ds\Big)^{1/2}$$

the bound (3.15) reads

$$|\lambda_k - \lambda| \leq \frac{\varepsilon_n(\lambda)}{1 - \bar{\varepsilon}_n(\lambda)} \tag{4.7}$$

provided that $\bar{\varepsilon}_n(\lambda) < 1$. Thus, this method leads to the parametric eigenvalue problem of finding $\lambda \in \mathbb{R}$ such that $\varepsilon_n(\lambda)$, i.e. the eigenvalue $\rho(\lambda)$, is locally minimized.

In [16] this approach has been applied to the operator Δ by taking trial functions $\varphi(x_1, x_2) = x_1^\nu x_2^\mu$.

Remark 5. We briefly compare the two methods. In both methods one has to determine local minima of a function $\varepsilon_n(\lambda)$ (cf. (4.3) or (4.6)). For the second method, only once must the $3(n+1)n/2$ integrals $\int_G \tilde{L}\varphi_\nu \tilde{L}\varphi_\mu \, dx$, $\int_G \tilde{L}\varphi_\nu \varphi_\mu + \varphi_\nu \tilde{L}\varphi_\mu \, dx$, $\int_G \varphi_\nu \varphi_\mu \, dx$, $\nu = 1(1)n$, $\mu = \nu(1)n$ and the $(n+1)n/2$ integrals $\int_{\partial G} \varphi_\nu \varphi_\mu \, ds$ be calculated (within a sufficient accuracy). Then for every λ the smallest eigenvalue of (4.5) has to be found.

Clearly by taking appropriate discretizations of G and ∂G the semi-infinite optimization problem (4.2) of the first method can be replaced by a finite problem. This method is particularly suitable when using trial functions $\varphi(\lambda, x)$ satisfying $\tilde{L}\varphi - \lambda\varphi = 0$ (For the construction of such functions φ for operators L other than the Laplace operator, see [21]).

In the case of $L = -\Delta$ such functions can be adjusted to special regions G to give extremely good approximations of eigenfunctions and eigenvalues with a moderate number of trial functions (cf. for example [13]).

In the second method the use of such functions $\varphi(\lambda, x)$ depending on λ would require the calculation of all $2(n+1)n$ integrals anew for every λ, which makes this method too expensive in that case. On the other hand, in the second method there does not arise, in principle, any additional problem when dealing with operators L other than $-\Delta$.

In both methods one can take advantage of the fact that one is solving a parametric problem, which means, roughly speaking, that one has to solve many neighboring problems.

Now we want to use the bounds in Theorem 4 given by the help of the Rayleigh quotient.

Let an approximation (λ^*, u^*) of an eigenpair be given, where $u_n^* = u_n^*(\alpha, x)$ has been calculated via the optimization problem (4.2). For shortness we restrict ourselves to the case where the functions $u(\alpha, x)$ satisfy the equation $\tilde{L}u - \lambda u = 0$. Then the constraints for $x \in G$ are eliminated, and (4.3) reads

$$|\lambda_k - \lambda^*| \leq \frac{\lambda^* \mu^*}{1 - \mu^*} =: \varepsilon. \tag{4.8}$$

To apply Theorem 4a we have to solve the boundary value problem (2.6), i.e. (3.5).

To this end define a space

$$W_n = \{w_n(\beta, x) = \sum_{\nu=1}^{N} \beta_\nu \psi_\nu(x),\ \beta = (\beta_1, \ldots, \beta_N)\} \tag{4.9}$$

with functions $\psi_\nu \in D(\tilde{L}) \setminus D(L)$. Here again for brevity we assume that $\tilde{L}\psi_\nu = 0$, $\nu = 1(1)N$ holds.

Then (3.5) could be solved approximately with the help of the problem ($N \in \mathbb{N}$ fixed)

$$RP_N\colon \ \underset{(\beta,\kappa)}{\text{Min}}\ \tilde{l}(\beta, \kappa) = \kappa, \qquad (\beta, \kappa) \in \mathbb{R}^{N+1}$$

subject to (4.10)

$$|w_N(\beta, x) - u_n^*(\alpha, x)| \leqq \kappa \qquad \text{for all } x \in \partial G.$$

Now, with a solution (β, κ), we set $\tilde{w} := w_N(\beta, \bullet)$, $\tilde{v}_1 := u^* - \tilde{w}$ and (cf. (2.20) with $r = 0$)

$$\tilde{\rho}_1 := \lambda^* \left(1 + \frac{\langle \tilde{w}, \tilde{v}_1 \rangle}{\langle \tilde{v}_1, \tilde{v}_1 \rangle}\right), \tag{4.11}$$

and by (2.17) and (2.22) we get the bound

$$|\lambda_k - \tilde{\rho}_1| \leqq \frac{4}{d(\rho_1)} \varepsilon^2 + |\rho_1 - \tilde{\rho}_1|. \tag{4.12}$$

For the difference $|\rho_1 - \tilde{\rho}_1|$, by (2.23) and (3.7) the explicit bound

$$|\rho_1 - \tilde{\rho}_1| \leqq \kappa \lambda^* \frac{1 + 4(\mu^* + \kappa) + \dfrac{\kappa}{1 - \kappa - \mu^*}}{1 - \mu^* - 2\kappa} \tag{4.13}$$

is valid provided that $\mu^* + 2\kappa < 1$.

Remark 6. Similarly the idea of calculating Rayleigh quotients applies when we have an approximate eigenpair given through the second method above. Here the boundary value problem (3.5) may be solved via the optimization problem

$$RQ_N\colon \ \underset{\beta}{\text{Min}} \int_{\partial G} (w_N(\beta) - u^*)^2 \, ds$$

with trial functions $w_N(\beta)$ from W_N (cf. (4.9)), leading to a linear equation.

The first method for solving the eigenvalue problem will be illustrated by the problem

$$\Delta u + \lambda u = 0 \quad \text{on} \quad G$$
$$u = 0 \quad \text{on} \quad \partial G$$

with the L-shaped region G indicated in Fig. 1.

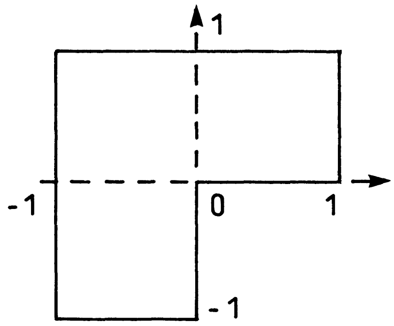

Fig. 1

As trial functions we choose

$$u_n(\lambda, \alpha, x) = \sum_{\nu=1}^{n} \alpha_\nu J_{\frac{2}{3}\nu}(\sqrt{\lambda} r) \sin \frac{2}{3} \nu \varphi \tag{4.14}$$

((r, φ) polar coordinates) satisfying $\Delta u_n(\lambda, \alpha, x) + \lambda u_n(\lambda, \alpha, x) = 0$ all $\lambda > 0$, $x \in G$; $\alpha \in \mathbb{R}^n$.

In [12] the first fifteen eigenvalues of the symmetric L-membrane have been calculated within a bound $|\lambda_k - \lambda| \leqq 10^{-7}$, only with the help of the parametric problem $P_n(\lambda)$, by applying the following algorithm.

Step 1. Compute the local minima of $\varepsilon_n(\lambda)$ on the discrete set $S = \{\lambda = 7, 8, \ldots, 100\}$ for a moderate number n of trial functions ($n \sim 20$).

To get better approximations for an eigenvalue λ_k, choosing an increment Δn and an error bound δ we proceed with

Step 2. Near a local minimum λ^* of $\varepsilon_n(\lambda)$ on S (corresponding to λ_k) calculate a local minimum $\lambda^{(n)}$ of $\varepsilon_n(\lambda)$. Set $\lambda^* := \lambda^{(n)}$.

Step 3. Increase the number of trial functions, i.e. set $n = n + \Delta n$ and compute a local minimum $\lambda^{(n)}$ of $\varepsilon_n(\lambda)$ near λ^* yielding a bound

$$|\lambda_k - \lambda^{(n)}| \leqq \varepsilon_n(\lambda^{(n)}).$$

Stop if $\varepsilon_n(\lambda^{(n)}) < \delta$. Otherwise, set $\lambda^* = \lambda^{(n)}$ and repeat Step 3.

To make use of the Rayleigh quotient, we have to solve via (4.10) the problem $\Delta w_N = 0$ on G, $w_N = u^*$ on ∂G. Appropriate trial functions in our situation are given by

$$w_N(\beta, x) := \sum_{\nu=1}^{N} \beta_\nu r^{\frac{2}{3}\nu} \sin \tfrac{2}{3} \nu \varphi \tag{4.15}$$

satisfying $\Delta w = 0$ on G. The bound (4.12) suggests the following modification of the algorithm above to locally improve the bound for eigenvalues.

Step 1'. The same as Step 1.

Step 2'. If necessary apply Steps 2 and 3 until a λ^* is found such that $4\varepsilon_n(\lambda^*)/d(\lambda^*)<1$.

Step 3', With the solution $u_n^*(\lambda^*, \alpha, \bullet)$ corresponding to $P_n(\lambda^*)$, solve RP_N (cf. (4.10)) to give $\tilde{\rho}_1$ (cf. (4.11)) and the bound (4.12),

$$|\lambda_k-\tilde{\rho}_1|\leqq\frac{4}{d(\rho_1)}\varepsilon_n^2(\lambda^*)+|\rho_1-\tilde{\rho}_1|=:\tau.$$

(Here N must be chosen large enough to garantee$|\rho_1-\tilde{\rho}_1|<4\varepsilon_n^2(\lambda^*)/d(\rho_1)$; for example $N=2n$). Stop if $\tau\leqq\delta$.

Step 4'. Set $n=n+\Delta n$, $\lambda^*=\tilde{\rho}_1$. Solve $P_n(\lambda^*)$ and goto Step 3'.

Table 1

Step	n	λ^* or $\tilde{\rho}_1$	Bound
1'	9	29.50	$0.86\cdot 10^{-1}$
3'		29.5216	$0.33\cdot 10^{-2}$
4'	12	29.5216	$0.61\cdot 10^{-3}$
3'		29.52148111	$0.23\cdot 10^{-6}$
4'	15	29.52148111	$0.10\cdot 10^{-4}$
3'		29.5214811141	$0.68\cdot 10^{-9}$

Table 2

Step	n	λ^* or $\tilde{\rho}_1$	Bound
1'	9	41.47	$0.64\cdot 10^{-0}$
3'		41.469	$0.19\cdot 10^{-0}$
4'	12	41.469	$0.25\cdot 10^{-1}$
3'		41.474491	$0.41\cdot 10^{-3}$
4'	15	41.474491	$0.11\cdot 10^{-3}$
3'		41.47450990	$0.32\cdot 10^{-7}$
4'	18	41.47450990	$0.23\cdot 10^{-5}$
3'		41.4745098902	$0.17\cdot 10^{-9}$

Remark 7.

a) The modified algorithm has the following advantage. After getting a first bound $|\lambda_k-\lambda^*|\leqq\varepsilon_n(\lambda)$ in Step 2', the execution of Step 3' demands the solution of the two optimization problems RP_N, $P_n(\lambda^*)$, which must be compared with the approximate computation of a local minimum of the function $\varepsilon_n(\lambda)$, requiring possibly many function evaluations of $\varepsilon_n(\lambda)$, i.e. solutions of the problems $P_n(\lambda)$.

b) Attention must be paid in Step 3' to the evaluation of

$$\tilde{\rho}_1=\lambda^*\left(1+\frac{\langle\tilde{w},\tilde{v}_1\rangle}{\langle\tilde{v}_1,\tilde{v}_1\rangle}\right).$$

For in (4.13) in our example, we have more precisely

$$|\rho_1 - \tilde{\rho}_1| \leqq \kappa \lambda^* \frac{1 + 4(\mu^* + \kappa) + \dfrac{\kappa}{1 - \kappa - \mu^*}}{1 - \mu^* - 2\kappa} + \tilde{\varepsilon}$$

where $\tilde{\varepsilon}$ is the error resulting from an inaccurate calculation of the integrals $\langle \tilde{w}, \tilde{v}_1 \rangle$, $\langle \tilde{v}_1, \tilde{v}_1 \rangle$.

Thus, these integrals must be computed within an accuracy such that $\tilde{\varepsilon}$ does not disturb the quality of the bound.

In our examples the integrals have been computed with the help of a two dimensional Romberg method after applying a transformation which transforms the integrands to functions analytic on all of $\mathbb{R}^2$.

Typically, the algorithm described above behaves as indicated in Tables 1, 2. We started with approximations $\lambda_4^* = 29.5$, $\lambda_6^* = 41.47$ of the eigenvalues λ_4, λ_6 of the L-shaped membrane in Step 3′. Both values were obtained by applying an appropriate interpolation to the values calculated in Step 1′, i.e. Step 2′ could be omitted. Tables 1, 2 give the bounds for the eigenvalues λ_4, λ_6 successively computed in Step 3′ or Step 4′ ($|\lambda_\nu - \tilde{\rho}_1|$, $|\lambda_\nu - \lambda^*| \leqq$ bound, $\nu = 4{,}6$). The step number in the first column of the tables refers to the substeps of the algorithms, whereas the numbers n in the second column indicate the number of trial functions in (4.14) used in the problems $P_n(\lambda)$ (cf. (4.2)).

Remark 8. In the computation we have made use of the fact that the eigenfunctions of the symmetric L-membrane either are symmetric or antisymmetric with respect to the angle $3\pi/4$. Thus, in (4.14) and (4.15) we only have to consider trial functions with ν odd in the symmetric or ν even in the antisymmetric case.

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