Math 116 Numerical PDEs: Homework 7

due Fri midnight, Nov 14

Here you implement the Method of Particular Solutions (MPS) for Ω the triangle with corners $(0,0)$, $(1,0)$ and $(1, \tan\beta)$ where β is the corner angle at the origin. First a BVP then an eigenvalue problem.

1. Consider the Fourier-Bessel basis $\phi_n(x) = J_n(kr)e^{in\theta}$, for $-N \le n \le N$, which satisfies the Helmholtz equation at wavenumber k. Use this to solve the interior Helmholtz BVP with Dirichlet data $f = u|_{\partial\Omega}$ from the exact solution $u(x) = 5iH_0(k|x - x_0|)$, with $x_0 = (0,2)$, and $k = 10$, as follows.

   (a) Make a function which returns the nth function in the set evaluated at a list of points `x` (a 2-by-m coordinate array), with wavenumber k. Try to vectorize it.

   (b) Make a list of $3M$ boundary quadrature nodes y_j and weights w_j using M-point Gaussian quadrature on each side of the triangle (duplication at corners is okay). Use your above function to fill columns of the matrix A, with elements $A_{j,n+N+1} = \sqrt{w_j}\phi_n(y_j)$. Fill a rhs vector with $\sqrt{w_j} f(y_j)$ and solve the least-squares linear system for the coefficients α_n.

   (c) Plot the resulting sum of basis functions over the domain. For $\beta = \pi/3$, $k = 10$, $M = 30$, and $N = 30$, plot a color image of $\log_{10}$ of the error over a rectangle enclosing the domain, showing the color range $[-16, 0]$. How does the interior error distribution compare to your BIE method?

   (d) With other parameters fixed, plot, and classify, the convergence of absolute error in the approximate solution at interior point $x = (0.7, 0.5)$ vs N.

   (e) What does the condition number do as N increases? How about the coefficient norm $\|\boldsymbol{\alpha}\|$? How come the solution reaches the accuracy it does?

2. For the eigenvalue problem you may keep most of the above code. Fix $M = 30$, $N = 20$, and $\beta = \pi/3$.

   (a) Set up 90 points z_j randomly chosen uniformly inside the triangle. Fill the matrix B with entries $B_{j,n+N+1} = \phi_n(y_j)$. Use the following Matlab code (also online) to compute the generalized eigenvalues μ and eigenvector matrix V of the matrix pair A^*A, B^*B:

   ```
   [V,D] = eig(B'*B); D = diag(D); i = find(D > 1e-14*max(D));
   V = V(:,i) .* repmat(1./sqrt(D(i)).', [size(B,2) 1]);
   F = V'*A'*A*V; [W mu] = eig((F+F')/2);     % projects out numerical Nul(B)
   mu = diag(mu); V = V*W;
   ```

   The lowest μ should be the first in the list, and its eigenvector in `V(:,1)`.

   (b) For $4 < k < 15$ plot the square-root of the minimum μ as a function of k. You should see ten local minima, giving the Dirichlet eigenvalues.

   (c) For $k = 5.54126$ reconstruct the eigenfunction using the coefficients in the eigenvector. Choose a color scale to illustrate its size inside the triangle.

   (d) BONUS: find the first few eigenvalues to high accuracy by locally minimizing the lowest generalized eigenvalue μ as a function of k.