Math 116 Numerical PDEs: Homework 6

due Fri midnight, Nov 7

This week you are coding! Questions 2–5 build on HW5, and on each other, in incremental steps, so always try to extend/tweak your existing code.

1. [Jon's question] Rewrite your code from HW4 #1 (a) so that it can graph any function of the form $u(x) = \int_a^b k(x-y)f(y)dy$ on any interval $[c,d]$.

2. Here you solve the interior Dirichlet BVP for Laplace's equation using the closed planar curve from last time with radial function in polar coordinates $R(t) = 1 + a\cos(3t)$, with $t \in [0, 2\pi)$, with $a = 0.3$.

   (a) Code the formula for curvature $\kappa(t)$ in terms of $R(t)$, $R'(t)$, $R''(t)$, and feed it the values for this radial function. Let t_j be nodes in the periodic trapezoid rule, as in HW5 #4. Plot $\kappa(t_j)$ for yourselves to check it gives what you expect. Make a function to fill the Nyström kernel matrix: diagonal elements via $\kappa(t_j)$, off-diagonal elements via your HW5 #4a function. Check the diagonal elements smoothly agree with those near the diagonal, by plotting a color image of the matrix; debug as needed.

   (b) Let's use a BVP with known solution $u(x_1, x_2) = \cos(x_1)e^{x_2}$ which you can check is harmonic. Then its Dirichlet boundary data is $f = u|_{\partial\Omega}$, which you should use to fill a column vector of values $f(y_j)$. Solve the linear system corresponding to $(I - 2D)\tau = -2f$ to get the τ vector at the nodes. Tweak your code from HW5 #4 to compute the resulting solution field $u^{(N)}$ over the grid in $[-1.3, 1.3]^2$ with spacing 0.02, for $N = 50$. Produce a 2D color image showing $\log_{10}|u^{(N)} - u|$ with a color scale range $[-16, 0]$. How does the error vary inside the domain?

   (c) For the fixed location $x = (0.2, 0.1)$, show convergence vs N of this error on an appropriate plot, and state the convergence order or rate. [Note you shouldn't evaluate over the whole grid here].

3. Now adjust your code to also solve the analogous BVP for the Helmholtz equation in the same Ω. [Warning: you are now using complex numbers, so to transpose an array you should use `.'` . Also graph-plotting only handles real-valued arrays.]

   (a) Compute the DLP kernel $\partial\Phi(x,y)/\partial n_y$ given $\Phi(x,y) = (i/4)H_0(k|x-y|)$; note $H_0'(z) = -H_1(z)$. Give your HW5 #4a kernel function an extra k input argument and make it switch between Laplace and Helmholtz (which has the same diagonal part) using `if k==0 ... else ... end`.

   (b) Fixing wavenumber $k = 6$, test with boundary data for the solution field $u(x) = 5iH_0(k|x - x_0|)$, with 'source point' $x_0 = (0, 2)$ which is outside Ω. Produce the same $\log_{10}$ error image as above. [Since special functions are slow computation will take about 5 sec].

   (c) What now is the convergence order, or rate, at the fixed location $x = (0.2, 0.1)$? Is it as good as before? What does this suggest about the smoothness of the DLP kernel?

4. Make your code switchable to the exterior Helmholtz BVP, which is as simple as changing the signs in the BIE to $(I + 2D)\tau = 2f$.

   (a) Use this to solve for the scattered field $u = u^i + u^s$ due to the incident plane wave $u^i(x) = e^{ik\hat{n}\cdot x}$ with wavenumber $k = 10$ and direction $\hat{n} = (\cos 0.2, \sin 0.2)$ reflecting from the domain Ω from before with Dirichlet boundary condition. Produce a 2D color image showing u over the region $[-4, 4]^2$; this should look familiar from the course flyer. In particular check by eye that $u_\partial\Omega$ is zero. BONUS: set the values inside Ω to zero since they are not physically relevant.

(b) Since the solution is not known analytically, observe the convergence with N until you are confident in the first 4 significant digits of the total field u at the point $(-2, 0)$ and quote them.

5. Here you explore the dependence on wavenumber k of the above Helmholtz kernel matrices—this will be quick since you already have a function for these.

   (a) Plot a graph vs $k \in [0.1, 4]$ of the lowest singular value of the Nystroöm matrix for $(I - 2D)$ you used for the interior BVP. Locate as accurately as you can the lowest k where its condition number blows up.

   (b) By choosing generic boundary data f, answer whether the actual physical solution to the interior BVP blows up near this k, or if it is merely a numerical effect.

   (c) Plot a similar graph for the matrix $(I + 2D)$ which you used in the exterior BVP. Locate as accurately as you can the lowest nonzero k where its condition number blows up.

   (d) Answer as before whether the physical solution to the exterior BVP blows up near this k, or if it is merely a numerical effect?

   Note: the k values you found in (a) are the Dirichlet eigenvalues of Ω, and in (c) the Neumann eigenvalues.