Math 116 Numerical PDEs: Homework 5

due Fri midnight, Oct 31

1. Basic fitting of convergence rates (easy). Say an exponentially-convergent numerical procedure gives error $E_n = 10^{-5}$ at $n = 10$ and $E_n = 10^{-12}$ at $n = 20$. Assuming it is in its asymptotic regime, find K such that $E_n = cK^{-n}$. Find α such that $E_n = ce^{-\alpha n}$. [Please use either number to discuss rates from now on; both have their merits].

2. (a) If A is a bounded operator, and B a compact operator, prove that AB is compact.

   (b) Prove that $C([0, 1])$ is not a *complete* space in the L^2 norm. [Hint: construct a sequence (u_n) of steeper and steeper functions which you can show are Cauchy convergent, but whose limit is not continuous]

3. Continuation of last question of HW4.

   (a) Use your answer to b) from that question to write an explicit formula for the solution u to $(I-K)u = f$ in terms of the function f and the coefficients $\{k_m\}$, and massage it into the form of an integral operator with new kernel acting on f. Congratulations: you've just used the 'spectral representation' to write the kernel of $(I - K)^{-1}$!

   (b) What is the condition on $\{k_m\}$ such that $(I - K)^{-1}$ exists (and is bounded)?

   (c) Say the coefficients obey $k_m = O(|m|^{-1})$ as $|m| \to \infty$. What can you say about compactness of K in $L^2[0, 2\pi)$? [Hint: construct a sequence (K_n) of finite-dimensional approximations to K and consider the norm of $K_n - K$].

   (d) BONUS: Find the weakest condition you can on $\{k_m\}$ that implies compactness of K in $L^2[0, 2\pi)$. Discuss the consequences for the possible limit points of the eigenvalues λ_m as $|m| \to \infty$.

4. The fundamental solution for Laplace's equation in 2D is $\Phi(x, y) = -(1/2\pi) \ln |x - y|$. Here you set up essential machinery for double-layer representations.

   (a) Make a function which returns the normal derivative $\partial\Phi(x, y)/\partial n_y$ given vectors $x, y \in \mathbb{R}^2$ and the unit vector $n_y \in \mathbb{R}^2$. Generalize your routine so that it handles multiple x vectors (*e.g.* a 2-by-n matrix of coordinates of n such vectors), and returns the corresponding list of outputs. [Be sure to test it on known inputs].

   (b) Use the above to produce a contour plot of $\partial\Phi(x, y)/\partial n_y$ for $y = 0$, $n_y = (1, 0)$, for x in the square $[-1, 1]^2$. This should be a 3-line program.

   (c) Consider the circle Γ defined by $y(t) = (\cos t, \sin t)$ for $t \in [0, 2\pi)$, on which arclength is simply t. Use periodic trapezoidal quadrature with $n = 20$, and your above function, to write a code which approximates the boundary integral

   $$u(x) = \int_0^{2\pi} \frac{\partial\Phi(x, y(t))}{\partial n_{y(t)}} dt \tag{1}$$

   for a set of points x. Note this is the double-layer operator, $u = D\tau$, acting on the function $\tau \equiv 1$. Use this to make a 3D plot of $u(x)$, for x in the square $[-1, 1]^2$. Check that interior values are roughly -1, and exterior zero.

(d) Use the change-of-variables formula to generalize your above code to the closed curve Γ defined by $y(t) = (R(t)\cos t, R(t)\sin t)$ where $R(t) = 1 + a\cos(3t)$ is a periodic polar function. Choose the 'wobbliness parameter' $a = 0.3$. [Hint: you'll need to figure out the 'speed function' $|y'(t)|$]. Thus make a 3D plot of

$$u(x) = \int_\Gamma \frac{\partial \Phi(x, y(t))}{\partial n_{y(t)}} ds_y \tag{2}$$

which is again $u = D\tau$ for $\tau \equiv 1$, over x in the square $[-2, 2]^2$.

(e) The above plot should approximate -1 inside Γ; make a contour plot of $\log_{10}$ of the absolute deviation from this value over the interior of Γ. How does the error seem to vary in the domain?

(f) For the fixed location $x = (0.2, 0.1)$, show convergence vs n of this error on an appropriate plot, and state the convergence order or rate. What n is needed to reach the minimum error? BONUS: How does the above depend on the choice of point x?