Math 116 Numerical PDEs: Homework 4

due Fri midnight, Oct 24

1. [Jon's question]

   (a) Plot the graph of the function $x \mapsto \int_{[-1,1]} \sqrt{1-y^2} e^{x-y^2} dy$ on the interval $[0,1]$.

   (b) Prove that all of the roots of polynomial which defines the nodes for $(n+1)$-node Gaussian quadrature are simple. [Hint: Assume that the roots are not unique. Glancing at the proof showing you can't integrate exactly all polynomials of degree $2n+2$ may help.]

2. Solve analytically the rank-1 second-kind integral equation,

$$u(s) + \int_0^1 st^3 u(t) dt \;=\; 1, \qquad \text{for } s \in [0,1] \tag{1}$$

   [Hint if stuck: u is the RHS plus something in the range of K, the integral operator]. Compute $\|K\|_\infty$. Is K compact? (why?)

3. Code up the 1D Nyström method in a way that allows you to switch easily between different quadrature schemes (*e.g.* by setting a switch variable at the start of your code). Apply it to the second-kind Fredholm equation

$$e^s + \int_0^1 e^{st} u(t) dt \;=\; e^s + \frac{1}{s+1}(e^{s+1} - 1) \tag{2}$$

   which you can check has exact solution $u(t) = e^t$.

   (a) Produce plots that show the convergence vs N, the number of nodes, of the maximum error magnitude in u over the nodes, for the two schemes: i) composite trapezoid, and ii) Gaussian quadrature. Categorize the convergence in each case and relate it to that of the quadrature scheme. What N is required in each case to reach an error smaller than 10^{-5}?

   (b) How does the condition number of the linear system you are solving change with N? (You don't need to plot this, just describe).

   (c) At $N = 5$ for Gaussian quadrature, produce a plot of the difference between the Nyström solution for $u(t)$ and the exact solution, over the interval $[0,1]$. (Don't show the two functions, just subtract them). Overlay the 5 nodes onto your graph. Is the true error sup norm of the solution reflected by the maximum error magnitude in u over the nodes, as you assumed in the previous part?

4. Naively adjust your code to attempt to solve the first-kind Fredholm equation on the periodic interval $[0, 2\pi)$,

$$\int_0^{2\pi} e^{a\cos(s-t)} u(t) dt \;=\; 2\pi I_0\left(\sqrt{1 + 2a\cos(s) + a^2}\right) \tag{3}$$

   where $I_0(\cdot)$ is the modified regular Bessel function of order zero (see Matlab's `besseli(0,...)` or Python's `scipy.special.iv(0,...)`). This has the exact solution $u(t) = e^{\cos(t)}$, trust me. However, such deconvolution problems are *ill-posed* ! (infinite condition number, *i.e.* u is arbitrarily sensitive to changes in f). Nevertheless, attempt to use our preferred quadrature scheme for smooth periodic functions.

(a) Choose the value $a = 0.5$ and plot the sup norm of the solution error at the nodes, as a function of $N = 2, 3, \ldots, 30$, choosing axes which show the behavior.

(b) Explain why the convergence behavior eventually does what it does. [Hint: see part b of previous question]. This shows the problem with first-kind IE's when no *regularization* is used. In contrast, second-kind are always stable.

5. Here you explore analytically how Fredholm equations with convolution kernels, that is kernels of the form $k(s,t) = k(t-s)$ on the interval $[0, 2\pi)$, where $k : \mathbb{R} \to \mathbb{C}$ is a 2π-periodic function, become trivial in the Fourier basis.

(a) Show that e^{imt}, $m \in \mathbb{Z}$, is an eigenfunction of any convolution operator K (*i.e.* integral operator with convolution kernel k), and find its eigenvalue λ_m.

(b) By writing $f(s) = \sum_{m\in\mathbb{Z}} f_m e^{-ims}$ and similar for u and k, convert the first-kind Fredholm equation $Ku = f$ into a set of relations involving the Fourier representations $\{f_m\}$, $\{u_m\}$ and $\{k_m\}$. [Hint: you'll need orthogonality of $\{e^{imt}\}$ on $[0, 2\pi)$]

(c) Thus use Parseval's equality to find a formula for $\|K\|_2$, *i.e.* the operator norm from $L^2([0, 2\pi))$ to itself, and express boundedness as a condition on the set $\{k_m\}$.

(d) BONUS: Say k is square-integrable on $[0, 2\pi)$. Can you prove something about the boundedness of K ? Can you do the same for K^{-1} ?