Math 113
Homework Assignment Number Two
Due Wednesday, January 20^{th}

INSTRUCTIONS: As usual, for the "true/false" questions, just circle the correct answer. No justifications are required, but don't guess. You score is based on #right minus #wrong.

1. **TRUE or FALSE**: Every finite dimensional normed vector space is a Banach space.

2. **TRUE or FALSE**: Every compact Hausdorff space is a normal topological space.

3. **TRUE or FALSE**: If W is a closed subspace of a normed vector space V, then the quotient map $q : V \to V/W$ has norm one provided $W \neq V$.

4. **TRUE or FALSE**: Suppose that V and W are normed vector spaces and that $T : V \to W$ is linear. If V is finite dimensional, then T is bounded.

5. **TRUE or FALSE**: Let $C^1_{\mathbf{R}}([0,1])$ be the set of real-valued functions on $[0,1]$ with a continuous derivative on $[0,1]$. Let $\|f\| := \|f\|_\infty + \|f'\|_\infty$. Then $C^1_{\mathbf{R}}([0,1])$ is a Banach space with respect to $\|\cdot\|$.[1]

6. Work E.1.2.9 in the text.

ANS: Let (X,d) be a metric space. Fix $x \in X$. Let $B_\epsilon(x) = \{\, y \in X : d(x,y) < \epsilon \,\}$. I claim $\mathcal{N} = \{\, B_{\frac{1}{n}}(x) \,\}_{n=1}^\infty$ is a neighborhood base at x. Let V be a neighborhood of x. Then there is an $\epsilon > 0$ such that $x \in B_\epsilon(x) \subseteq V$. Choose n such that $\frac{1}{n} < \epsilon$. Now $x \in B_{\frac{1}{n}}(x) \subseteq B_\epsilon(x) \subseteq V$. This proves the claim.

Next suppose that (X,d) is second countable; let $\{\, A_n \,\}_{n=1}^\infty$ be a basis for the topology. For each $n \geq 1$, choose any $x_n \in A_n$. I claim that $D = \{\, x_n \,\}_{n=1}^\infty$ is dense in X, and hence, that X is separable. But if V is a nonempty open subset of X, say $x \in V$, then there is a n such that $x \in A_n \subset V$. Thus, $D \cap V \neq \emptyset$. The claim follows.

[1] You may want to use the result that if $f_n \to f$ uniformly on $[0,1]$ and each f_n is differentiable with $f'_n \to g$ uniformly on $[0,1]$, then f is differentiable and $f' = g$. A proof of this statement follows easily from Theorem 7.17 of Rudin's *Principles of Real Analysis.*

Now suppose that (X, d) is separable; suppose that $D = \{ x_n \}_{n=1}^{\infty}$ is dense. Let $\rho = \{ B_{\frac{1}{m}}(x_n) : n, m \geq 1 \}$. I will show that ρ is a basis and then it follows by definition that X must be second countable. Let V be open in X and suppose that $x \in V$. Then there exists $\epsilon > 0$ such that $B_\epsilon(x) \subseteq V$. Choose m so that $\frac{1}{m} < \frac{\epsilon}{2}$. Since D is dense, there is a x_n such that $d(x_n, x) < \frac{1}{m}$. Then if $y \in B_{\frac{1}{m}}(x_n)$, we have $d(y, x) \leq d(y, x_n) + d(x_n, x) < \frac{1}{m} + \frac{1}{m} < \epsilon$. Thus,

$$x \in B_{\frac{1}{m}}(x_n) \subseteq B_\epsilon(x) \subseteq V.$$

This establishes the claim.

7. Work E.2.1.1 in the text.

ANS: The interesting part of this problem is to show that "absolute convergence implies convergence" implies completeness.

Suppose that $\{ x_n \}_{n=1}^{\infty}$ is a Cauchy sequence. We accept that it suffices to show that $\{ x_n \}_{n=1}^{\infty}$ has a convergent subsequence. Using the definition of Cauchy sequence, an induction argument shows that there is a subsequence $\{ x_{n_k} \}_{k=1}^{\infty}$ such that

$$\|x_{n_{k+1}} - x_{n_k}\| < \frac{1}{2^k}.$$

Let $y_1 := x_{n_1}$, and if $k \geq 2$, let $y_k := x_{n_{k+1}} - x_{n_k}$. Then $\sum y_k$ is absolutely convergent. By assumption, there is a $x \in X$ such that

$$x = \lim_k \sum_{i=1}^{k} y_i = \lim_k x_{n_k}.$$

That's what we wanted.

8. Define two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a vector space V to be equivalent in they determine the same topology on V. Prove that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent if and only if there are nonzero positive constants c and d such that

$$c\|v\|_1 \leq \|v\|_2 \leq d\|v\|_1 \quad \text{for all } v \in V.$$

ANS: This is similar to a result proved in lecture. Let T be the identity map from $(V, \|\cdot\|_1) \to (V, \|\cdot\|_2)$. If the norms are equivalent, then T is continuous and therefore bounded. Hence there is a $d \geq 0$ such that $\|x\|_2 = \|Tx\|_2 \leq d\|x\|_1$ for all x. Since $\|\cdot\|_2$ is a norm, $d > 0$. Since T^{-1} is also continuous, there is a $c > 0$ such that $\|x\|_1 \leq \frac{1}{c}\|x\|_2$ for all x. This is what we wanted to prove.

If the norm inequalities hold, then T and T^{-1} are bounded maps — therefore they are continuous with respect to the topologies induced by the norms. In other words, the identity map is a homeomorphism. Therefore the topologies coincide.

9. (After Monday's lecture): Suppose that X is a locally compact Hausdorff space. Show that $C_0(X)$ is closed in $C^b(X)$ and that $C_c(X)$ is dense in $C_0(X)$.

ANS: Suppose that $f_n \to f$ with each $f_n \in C_0(X)$. We want to see that $f \in C_0(X)$. Fix $\epsilon > 0$. It suffices to see that

$$\{ x : |f(x)| \geq \epsilon \}$$

is compact. Since it is clearly closed, it is enough to see that it is contained in a compact set. But we can choose n such that $\|f - f_n\|_\infty < \epsilon/2$. Then

$$\{ x : |f(x)| \geq \epsilon \} \subset \{ x : |f_n(x)| \geq \epsilon/2 \}.$$

Since the latter is compact, this suffices to show that $C_0(X)$ is closed in $C^b(X)$.

To see that $C_c(X)$ is dense in $C_0(X)$, let $f \in C_0(X)$. Fix $\epsilon > 0$. Then by assumption,

$$K := \{ x : |f(x)| \geq \epsilon \}$$

is compact. By the version of Urysohn's lemma we proved in lecture, there is a continuous function $\phi : X \to [0, 1]$ with compact support such that $\phi(x) = 1$ for all $x \in K$. Since $\phi \in C_c(X)$, so is the pointwise product ϕf. But

$$\|f - \phi f\|_\infty \leq \epsilon$$

(since $|f(x) - \phi(x)f(x)| = (1 - \phi(x))|f(x)| \leq |f(x)|$). This suffices.

10. Let X be a normed vector space and let $B = \{ x \in X : \|x\| \leq 1 \}$ be the *unit ball*. Show that if B is compact, then X is finite dimensional.[2] Since this is E.2.1.3 in the text, I was embarrassed not to be able to give a "quick" proof. You can either follow my steps below, or provide a better proof yourself.[3]

(a) Let $V = \{ x \in X : \|x\| < 1 \}$ be the open unit ball. Show that there is a finite set $\{ x_1, \dots, x_n \} \subset X$ such that

$$B \subset \bigcup_{i=1}^{n} x_i + \frac{1}{2}V.$$

(b) Let $Y = \operatorname{span}\{ x_1, \dots, x_n \}$ and conclude that

$$V \subset Y + \frac{1}{2}V$$

[2] It is easy to go from here to showing that any normed vector space that is locally compact is necessarily finite dimensional.

[3] In fact, the "steps below" aren't my original ones. An student in the 2007 instance of this course, Chor Lam, came up with the "improved" version here. Can you find a better one?

(c) Let

$$Z := \bigcap_n (Y + \frac{1}{2^n} V).$$

Observe that $V \subset Z$ and prove that $Z = Y$.

(d) Conclude that $Y = X$.

ANS: Actually, Chor's solution was a bit cleaner than mine. (Well, his idea was cleaner, his execution left a bit to the imagination.)

Once you have $V \subset Y + \frac{1}{2}V$, we have $V \subset Y + \frac{1}{2}(Y + \frac{1}{2}V) = Y + \frac{1}{4}V$. Iterating,

$$V \subset Z := \bigcap_n (Y + \frac{1}{2^n} V).$$

Clearly, $Y \subset Z$. But it $z \in Z$, then there are $y_n \in Y$ such that $\|z - y_n\| < \frac{1}{2^n}$. But then $y_n \to z$, and since Y is closed, $z \in Y$. Thus $Z = Y$, and we have

$$V \subset Y.$$

This implies that $Y = X$. Therefore X is finite dimensional (since Y is).

Remark. *I thought E.2.1.6, E.2.1.8, E.2.1.9 and E.2.1.10 all illustrated some interesting examples of Banach spaces, but I couldn't bear the thought of more to grade.*