(PRE-)HILBERT SPACES

MATH 113 - SPRING 2015

PROBLEM SET #7

Problem 1 (Gram-Schmidt orthonormalization)**.** Let $\mathcal{X} = \{x_n\}_{n\geq 0}$ be a countable family of linearly independent vectors in a Hilbert space. Prove the existence of a countable orthonormal family $\mathcal{Y} = \{y_n\}_{n\geq 0}$ such that

$$\mathrm{Span}(x_0, \ldots, x_p) = \mathrm{Span}(y_0, \ldots, y_p)$$

for all $p \geq 0$.

Problem 2 (Orthogonal polynomials)**.** Let I be an interval of $\mathbb{R}$ and $w : I \to \mathbb{R}$ a continuous positive function such that $x \mapsto x^n w(x)$ is integrable on I for any integer $n \geq 0$. Denote by $\mathcal{C}$ the set of continuous functions $f : I \to \mathbb{R}$ such that $x \mapsto f^2(x)w(x)$ is integrable. Finally, for f and g real-valued functions on I, we define

$$\langle f, g\rangle_w = \int_I f(x)g(x)w(x)\,dx$$

1. Verify that $\mathbb{R}[X] \subset \mathcal{C}$ and that $\langle\cdot,\cdot\rangle_w$ is an inner product on $\mathcal{C}$. Denote by $\|\cdot\|_w$ the corresponding norm. Is $(\mathcal{C}, \|\cdot\|_w)$ a Hilbert space?

2. Prove the existence of an orthonormal basis $\{P_n\}_{n\geq 0}$ of $\mathbb{R}[X]$ such that the degree of P_n is n and its leading coefficient γ_n is positive.

3. Verify that the polynomials P_n satisfy a relation of the form

   $$P_n = (a_n X + b_n)P_{n-1} + c_n P_{n-2} \qquad (\dagger)$$

   and determine the sequences $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$ and $\{c_n\}_{n\in\mathbb{N}}$.

4. Prove that P_n has n distinct roots in I.

5. Assume I compact.

    (a) Find a constant C such that $\|f\|_w \leq C\|f\|_\infty$ for all $f \in \mathcal{C}$.

    (b) For f in $\mathcal{C}$, let $p_n(f)$ be the orthogonal projection of f on $\mathbb{R}_n[X]$. Prove that $p_n(f) \xrightarrow[n\to\infty]{\|\cdot\|_w} f$.

Hint: 1. You may choose a concrete w to study completeness. 3. Project (†) and express a_n in terms of γ_n and γ_{n-1}. 4. Compute $\langle P_n, \prod_\alpha (X-\alpha)\rangle_w$ where the product is taken over roots of P_n with odd order.

Problem 3. Let G be a group acting on a countable set X. Let $\mathcal{H} = \ell^2(X)$ be the Hilbert space of square-integrable functions on X for the counting measure.

1. Let A and B be subsets of X, with indicators denoted by χ_A and χ_B.

    (a) Give a condition on A, equivalent to $\chi_A \in \mathcal{H}$.

    (b) Give a condition on A and B, equivalent to $\chi_A \perp \chi_B$ in $\mathcal{H}$.

2. For $f \in \mathcal{H}$ and $g \in G$, define $\pi(g)f = x \mapsto f(g^{-1}\cdot x)$.

    (a) Prove that each $\pi(g)$ is a unitary operator on $\mathcal{H}$.

    (b) Prove that $\pi : G \longrightarrow \mathrm{U}(\mathcal{H})$ is a group homomorphism.

From now on, we assume that for every $x \in X$, the G-orbit $\{g\cdot x\ ,\ g \in G\}$ is infinite.

3. Let $A \subset X$ be such that $\chi_A \in \mathcal{H}$ and denote by C be the closure of the convex hull[1] of $C_0 = \{\pi(g)\chi_A\ ,\ g \in G\}$.

    (a) Prove the existence of a unique element ξ of minimal norm in C.

    (b) Verify that C is stable by each of the operators $\pi(g)$.

    (c) Prove that $\pi(g)\xi = \xi$ for all $g \in G$.

    (d) Deduce that ξ is constant on each G-orbit and conclude.

4. Let A, B be non-empty finite subsets of X and assume that $(g\cdot A)\cap B \neq \varnothing$ for all g in G.

    (a) Prove that $\langle f, \chi_B\rangle \geq 1$ for all $f \in C$.

    (b) Apply the previous result to ξ and conclude.

[1] the convex hull of a set S is the family of all possible convex combinations of elements of S.