C*-ALGEBRAS

MATH 113 - SPRING 2015

PROBLEM SET #6

Problem 1 (Positivity in C*-algebras)**.** The purpose of this problem is to establish the following result:
Theorem. *Let $\mathcal{A}$ be a unital* C**-algebra. For $a \in \mathcal{A}$, the following statements are equivalent.*

(a) *a is hermitian and $\mathrm{Sp}_{\mathcal{A}}(a) \subset [0, \infty)$*

(b) *There exists b in $\mathcal{A}$ such that $a = b^*b$*

(c) *There exists b hermitian in $\mathcal{A}$ such that $a = b^2$*

An element satisfying **(a)** *is said* ***positive****and we write $a \geq 0$.*

1. What are the positive elements in $\mathbb{C}$? Verify that the theorem holds in this case.

2. Let $a \in \mathcal{A}$ be hermitian. Prove that there exist positive elements u, v in $\mathcal{A}$ such that $a = u - v$ and $uv = vu = 0$.

3. Let $a \geq 0$ in $\mathcal{A}$ and $n \in \mathbb{N}^*$. Prove the existence of $b \geq 0$ such that $a = b^n$.

4. Verify that **(a)** $\Rightarrow$ **(c)** $\Rightarrow$ **(b)** in the theorem.

5. We want to prove that the elements u, v and b in 2. and 3. are unique. Assume that $a = u' - v'$ with u', v' positive and $u'v' = v'u' = 0$.

   (a) Prove that $P(a) = P(u') + P(-v')$ for any polynomial P.
   (b) Let f be the function defined on $\mathbb{R}$ by $f(t) = \max(t, 0)$. Prove that $u = f(u') + f(-v')$.
   (c) Show that $f(u') = u'$ and $f(-v') = 0$ and conclude.

(d) Use a similar method to prove that the element b in 3. is unique.

Hints: 2.& 3. Functional Calculus, $t \mapsto \max(t, 0)$, $t \mapsto \max(-t, 0)$, $t \mapsto t^{\frac{1}{n}}$.
5.(a) Start with $P(t) = t^n$.
5.(b) Approach f uniformly on $\mathrm{Sp}(a) \cup \mathrm{Sp}(u') \cup \mathrm{Sp}(v')$ by polynomial functions.

Solution. 1. Non-negative real numbers.

2. Consider the continuous functions f and g defined on $\mathbb{R}$ by $f(t) = \max(t, 0)$ and $g(t) = \max(-t, 0)$. Since a is hermitian, $\mathrm{Sp}(a)$ is included in $\mathbb{R}$ so $f(a)$ and $g(a)$ are defined by the Functional Calculus and the Spectral Mapping Theorem implies that $u = f(a)$ and $v = g(a)$ are positive. Moreover, $f(t)-g(t) = t$ and $f(t)g(t) = 0$for all $t \in \mathbb{R}$ so $a = u-v$ and $uv = vu = 0$ since the Functional Calculus map is a morphism of algebras.

3. Similarly, consider $h(t) = t^{\frac{1}{n}}$ on $\mathbb{R}^+$ and verify that $h(a)$ is a solution.

4. **(a)** $\Rightarrow$ **(c)** follows from 3. with $n = 2$ and **(c)** $\Rightarrow$ **(b)** is tautological. A proof of the remaining implication can be found in Proposition 1.3.6 of Conway's book: *A Course in Abstract Analysis*.

5. (a) The condition $u'v' = v'u' = 0$ implies that $a^n = u'^n + (-v')^n$ for $n \in \mathbb{N}$. The result follows by linear combination.

   (b) The subset $S = \mathrm{Sp}(a) \cup \mathrm{Sp}(u') \cup \mathrm{Sp}(v')$ is a compact of $\mathbb{R}$. By Stone-Weierstrass, there is a sequence $\{P_n\}_{n\in\mathbb{N}}$ of polynomials that converges uniformly to f on S. By continuity of the functional calculus map and the definition of u, the sequence $\{P_n(a)\}_{n\in\mathbb{N}}$ converges to u in $\mathcal{A}$. Since $P_n(a) = P_n(u') + P_n(-v')$, the result of the previous question implies that $u = f(u') + f(-v')$.

   (c) The relations $f(u') = u'$ and $f(-v') = 0$ directly follow from the definition of f. They imply that $u = u'$, which in turn implies that $v = v'$.

   (d) Assume that $a = b'^n$ and consider a sequence of polynomial functions Q_k converging to h uniformly on $\mathrm{Sp}(a) \cup \mathrm{Sp}(b')$. Passing to the limit in k in $Q_k(a) = Q_k(b'^n)$, we get $b = h(b'^n) = (h \circ k)(b')$ where $k(t) = t^n$ so that $h \circ k(t) = t$ on $\mathbb{R}^+ \supset \mathrm{Sp}(b')$ and $b = b'$.

□

Problem 2 (Non-commutative topology)**.** If $X \xrightarrow{\varphi} Y$ is a continuous map between two topological spaces, we denote by $\varphi^\sharp$ the map from $C(Y)$ to $C(X)$ defined by

$$\varphi^\sharp(f) = f \circ \varphi.$$

1. Prove that $X \mapsto C(X)$, $\varphi \mapsto \varphi^\sharp$ is a contravariant functor from the category of compact Hausdorff spaces with continuous maps to the category of commutative unital C*-algebras with $*$-morphisms.

2. What does the Gelfand-Naimark Theorem say about this functor?
   What more can be said?

A map φ between locally compact Hausdorff spaces X and Y is said **proper** if the inverse image of a compact in Y is a compact of X.

3. Show that C_0 is a contravariant functor from the category of locally compact Hausdorff spaces to the category of commutative C*-algebras. Specify the morphisms.

4. Prove that $C_0(X)$ is $*$-isomorphic to $C_0(Y)$ if and only if X and Y are homeomorphic.

5. Assume X compact and $X_0 \subset X$ open.

   (a) Prove that $C_0(X_0)$ is an ideal of $C(X)$.

   (b) Show that all ideals in $C(X)$ are of this form.

6. Complete the following 'dictionary' translating properties of topological spaces in terms of properties of algebras, commutative or not. You may restrict to the case of compact spaces whenever it makes sense.

Spaces	**Algebras**
...	unital
points	...
...	ideals
...	quotients
...	$*$-morphism
...	$*$-isomorphism
disjoint union	...
connected component	...

Hints: maximal ideals in $C(X)$ are of the form $\mathcal{J}_{x_0} = \{f \in C(X)\,,\, f(x_0) = 0\}$.
The spectrum of $C_0(X)$ is homeomorphic to X.
A **projection** in a C*-algebra is an element a that satisfies $a^2 = a^* = a$.

Solution. 2. The functor $X \mapsto C(X)$ is essentially surjective by the Gelfand-Naimark Theorem and it is clearly faithful. It is also full: given a $*$-morphism $\Phi : C(Y) \longrightarrow C(X)$, consider the map between the maximal ideal spaces $\cdot \circ \Phi : \Sigma_{C(X)} \longrightarrow \Sigma_{C(Y)}$ and use the homeomorphism $\Sigma_{C(X)} \simeq X$. To sum up, $C(\cdot)$ is a contravariant equivalence of categories between compact Hausdorff spaces and commutative unital C*-algebras.

3. The main point is that if f is in $C_0(Y)$ and φ is proper, then $f \circ \varphi$ is in $C_0(X)$. Consider, for $\varepsilon > 0$, a compact K_ε of Y outside of which $|f|$ does not exceed ε. Then, the same holds for $|f \circ \varphi|$ outside of $\varphi^{-1}(K_\varepsilon)$ which is compact by properness of φ.

4. One direction follows from the fact that $\Sigma_{C_0(X)}$ is homeomorphic to X. For the other, let $X \overset{\varphi}{\to} Y$ be a homeomorphism and verify that $\varphi^\sharp$ is a $*$-isomorphism.

5. (a) Let Y be the complement of X_0 in X. The kernel of the restriction morphism $f \mapsto f|_Y$ is exactly $C_0(X_0)$.

   (b) Let $\mathcal{J}$ be an ideal in $C(X)$. Then $C(X)/\mathcal{J}$ is a commutative unital C*-algebra, hence of the form $C(Z)$ for some compact Hausdorff space Z by Gelfand-Naimark. Let π denote the natural projection $C(X) \longrightarrow C(Z)$. As in 2., there exists a map $\rho : Z \longrightarrow X$ such that $\rho^\sharp = \pi$. The surjectivity of π implies the injectivity of ρ and $\mathcal{J} = \ker \pi \simeq C_0(Y)$ where Y is the complement of $\rho(Z)$ in X.

6. The last line in the table can be filled by remembering that a topological space X is connected if and only if any continuous function with values in $\{0, 1\}$ is constant and observing that a projection in $C(X)$ is such a function.

Spaces	**Algebras**
compact	unital
points	maximal ideals
open subsets	ideals
closed subsets	quotients
proper map	$*$-morphism
homeomorphism	$*$-isomorphism
disjoint union	direct sum
connected component	projection

$\square$