APPLICATIONS OF THE ARZELÀ-ASCOLI AND THE BAIRE CATEGORY THEOREMS

MATH 113 - SPRING 2015

PROBLEM SET #2

Problem 1 (Hölder maps)**.**
A function $f \in C([0,1],\mathbb{R})$ is said to be α-***Hölder*** if

$$h_\alpha(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha}$$

is finite. For $M > 0$ and $0 < \alpha \leq 1$, denote

$$H_{\alpha,M} = \{f \in C([0,1],\mathbb{R})\,,\ h_\alpha(f) \leq M \text{ and } \|f\|_\infty \leq M\}\,.$$

Prove that $H_{\alpha,M}$ is compact in $(C([0,1],\mathbb{R}), \|\cdot\|_\infty)$.

Solution. The Arzelà-Ascoli Theorem implies that it suffices to check that $H_{\alpha,M}$ is closed, bounded and equicontinuous. The set in question is the intersection of the closed ball $B_c(0,M)$ and $F = \{f \in C([0,1])\,,\ h_\alpha(f) \leq M\}$, so it is automatically bounded and it is enough to check that F is closed. To do so, consider a sequence $\{f_n\}$ of functions in F, that converges to f in $C([0,1])$. The pointwise convergence of the sequence implies that $\frac{|f(x)-f(y)|}{|x-y|^\alpha} \leq M$ for every $x \neq y$ so F is closed. To establish equicontinuity, let $\varepsilon > 0$ and verify that $\delta = \left(\frac{\varepsilon}{M}\right)^{\frac{1}{\alpha}}$ is an appropriate modulus of continuity. $\square$

Problem 2. Show that a normed linear space over $\mathbb{R}$ that has a countable algebraic basis cannot be complete.

Solution. Let E be a normed space with an algebraic basis $\{e_i\}_{i\in\mathbb{N}}$ and $F_n = \mathrm{span}(e_1,\dots,e_n)$. Each F_n is finite-dimensional, hence closed. Moreover, if F_n contained an open ball of radius $r>0$ it would also contain $B(0,r)$, which generates E, so E would be contained in F_n. Therefore, each F_n has empty interior and Baire's Theorem ensures that $\bigcup_{n\geq 1} F_n$ has empty interior too, which contradicts the fact that $\bigcup_{n\geq 1} F_n = E$. □

Problem 3. Let $f : [0,+\infty) \longrightarrow \mathbb{R}$ be continuous and assume that for all $x>0$,

$$\lim_{n\to\infty} f(nx) = 0.$$

Prove that $\lim\limits_{x\to\infty} f(x) = 0$.

Hint: for $\varepsilon > 0$ and $n\in\mathbb{N}$, consider $F_{n,\varepsilon} = \{x \geq 0\ ,\ \forall p \geq n\ ,\ |f(px)| \leq \varepsilon\}$.

Solution. Each $F_{n,\varepsilon}$ is closed as the intersection of inverse images of the closed subset $[0,\varepsilon]$ of $\mathbb{R}$ by the continuous functions $f(p\,\cdot)$ for $p\in\mathbb{N}$, $p\geq n$. The hypothesis on f implies that $(0,+\infty)\subset\bigcup_{n\geq 1} F_n$. Being locally compact, $(0,+\infty)$ is a Baire space so that there exists $n_0\in\mathbb{N}$ such that $\mathring{F}_{n_0}\neq\varnothing$.
In other words, there exist $0<\alpha<\beta$ such that $(\alpha,\beta)\subset F_{n_0}$, which means that

$$\forall x\in(\alpha,\beta)\,,\ \forall p\geq n_0 \quad , \quad |f(px)|\leq\varepsilon.$$

The result then follows from the fact that, for p large enough, the intervals $(p\alpha, p\beta)$ overlap. More precisely, the condition $(p+1)\alpha < p\beta$ is equivalent to $p > \frac{\alpha}{\beta-\alpha}$ so that if $N > \max(n_0, \frac{\alpha}{\beta-\alpha})$, one has $|f(x)|\leq\varepsilon$ for x in $\bigcup\limits_{p\geq N}(p\alpha,p\beta) = (N\alpha,+\infty)$. □

Problem 4. Show that nowhere differentiable functions are dense in $E = C([0,1],\mathbb{R})$ equipped with its ordinary norm.

Hint: consider, for $\varepsilon > 0$ and $n \in \mathbb{N}$,

$$U_{n,\varepsilon} = \left\{ f \in E \, , \, \forall x \in [0,1] \, , \, \exists y \in [0,1] \, , \, |x-y| < \varepsilon \quad \text{and} \quad \left| \frac{f(y)-f(x)}{y-x} \right| > n \right\}.$$

Solution. We first prove that each set $U_{n,\varepsilon}$ is open because its complement $U^c_{n,\varepsilon}$ is closed. Observe that

$$U^c_{n,\varepsilon} = \left\{ f \in E \, , \, \exists x \in [0,1] \, , \, \forall y \in [0,1] \, , \, |x-y| < \varepsilon \Rightarrow \left| \frac{f(y)-f(x)}{y-x} \right| \le n \right\}.$$

and let $\{f_k\}$ be a sequence in $U^c_{n,\varepsilon}$ that converges to f in E. For each k, there exists $x_k \in [0,1]$ such that $|x_k - y| < \varepsilon \Rightarrow \left| \frac{f(y)-f(x_k)}{y-x_k} \right| \le n$. Since $[0,1]$ is compact, $\{x_k\}$ has a convergent subsequence $\{x_{\varphi(k)}\}$. Denote x its limit and let y in $[0,1]$ be such that $0 < |x-y| < \varepsilon$. For k large enough, one has $0 < |x_{\varphi(k)} - y| < \varepsilon$ so that $\left| \frac{f_{\varphi(k)}(y) - f_{\varphi(k)}(x_{\varphi(k)})}{y - x_{\varphi(k)}} \right| \le n$ and the uniform convergence $f_{\varphi(k)} \to f$ implies that $\left| \frac{f(y)-f(x)}{y-x} \right| \le n$, so that f belongs to $U^c_{n,\varepsilon}$.

Now we prove that $U_{n,\varepsilon}$ is dense in E. Polynomials are dense in E, so it suffices to prove that functions of class C^1 can be approximated by elements of $U_{n,\varepsilon}$.
For $p \ge 1$ integer, let v_p be a continuous function on $[0,1]$, affine on each interval $\left[\frac{k}{2p}, \frac{k+1}{2p}\right]$ and such that $v_p\left(\frac{k}{2p}\right) = 0$ (resp. $= 1$) if k is even (resp. odd). Let f be a function of class C^1 on $[0,1]$ and $g_p = f + \lambda v_p$. By construction, $\|f - g_p\|_\infty \le \lambda$ so g_p can be chosen arbitrarily close to f.
If $x \ne y$ in $[0,1]$, then

$$\begin{aligned} \left| \frac{g_p(x)-g_p(y)}{x-y} \right| &\ge \lambda \left| \frac{v_p(x)-v_p(y)}{x-y} \right| - \left| \frac{f(x)-f(y)}{x-y} \right| \\ &\ge \lambda \left| \frac{v_p(x)-v_p(y)}{x-y} \right| - \|f'\|_\infty. \end{aligned}$$

Let $p > \frac{1}{2\lambda}(n + \|f'\|_\infty)$. For any $x \in [0,1]$, there exists $y \in [0,1]$ within ε of x and in the same interval $\left[\frac{k}{2p}, \frac{k+1}{2p}\right]$. By definition of v_p, the latter implies that

$\left|\frac{v_p(x)-v_p(y)}{x-y}\right| = 2p$. Then

$$\left|\frac{g_p(x)-g_p(y)}{x-y}\right| \geq 2p\lambda - \|f'\|_\infty > n$$

so that $g_p \in U_{n,\varepsilon}$.

The Baire Category Theorem ensures that $U = \bigcap_{n\geq 1} U_{\frac{1}{n},n}$ is dense in E. Let $f \in U$ and $x \in [0,1]$. Then there is a sequence $\{x_n\}$ such that $0 < |x_n - x| < \frac{1}{n}$ and $\left|\frac{f(x_n)-f(y)}{x_n-y}\right| > n$, which prevents f from being differentiable at x.

$\square$