# METRIC SPACES

## MATH 113 - SPRING 2015

## PROBLEM SET #1

**Problem 1** (Distance to a subset and metric Urysohn's Lemma).
Let $(E, d)$ be a metric space. For any subset $A \subset E$ and any point $x \in E$, the *distance* between $x$ and $A$ is defined by

$$d(x, A) = \inf_{a \in A} d(x, a).$$

1. Show that $d(x, A) = d(x, \bar{A})$.

2. Show that $d(\cdot, A)$ is 1-Lipschitz.

3. Let $A$ and $B$ be disjoint closed subsets of $E$. Prove the existence of a continuous function $f : E \longrightarrow \mathbb{R}$ such that:

   (a) $0 \leq f(x) \leq 1$ for all $x \in E$;
   (b) $f(x) = 0$ for all $x \in A$;
   (c) $f(x) = 1$ for all $x \in B$.

**Problem 2** (Completeness is not a topological property).
Let $E = (0, +\infty)$ and for $x, y \in E$, consider $\delta(x, y) = \left| \frac{1}{x} - \frac{1}{y} \right|$.

1. Prove that $\delta$ is a distance on $E$ and that it induces the same topology as the Euclidean distance $d$.

2. Is the map $x \mapsto x^{-1}$ uniformly continuous as a map from $(E, d)$ to itself ? As a map from $(E, d)$ to $(E, \delta)$?

3. Is $(E, \delta)$ complete ? What about $((0, 1], d)$ and $((0, 1], \delta)$ ?

**Problem 3** (The Banach Contraction Principle)**.**
Let $(E, d)$ be a complete metric space and $f : E \longrightarrow E$.

1. Show that if $f$ is $k$-Lipschitz with $k < 1$, the equation $f(x) = x$ has a unique solution in $E$.

2. Show that if $E$ is compact, it is enough to have $d(f(x), f(y)) < d(x, y)$ for all $x$, $y$ to obtain the same result.

**Problem 4** (Completeness of $\ell^2(\mathbb{N})$)**.**
Show that the set of sequences $U = \{u_n\}$ such that $\sum_{n \geq 0} |u_n|^2$ converges is complete for the norm $\|U\|_2 = \left( \sum_{n=0}^{\infty} |u_n|^2 \right)^{\frac{1}{2}}$

**Problem 5** (Cantor's Intersection Theorem)**.**
Let $(E, d)$ be a metric space and $A \subset E$ a non-empty subset. The *diameter* of $A$ is defined by

$$\operatorname{diam}(A) = \sup_{x,y \in A} d(x, y).$$

Prove that $E$ is complete if and only if for every decreasing sequence $\{F_n\}_{n \in \mathbb{N}}$ of closed subsets of $E$ such that $\lim_{n \to \infty} \operatorname{diam}(F_n) = 0$, there is a point $x$ such that

$$\bigcap_{n \in \mathbb{N}} F_n = \{x\}.$$

**Problem 6** (Characterizations of compactness for metric spaces)**.**
Let $(E, d)$ be a metric space. Prove that the following conditions are equivalent.

(i) $E$ has the *Borel-Lebesgue* property, *i.e.* is topologically compact.

(ii) If $\mathcal{F}$ is a family of closed subsets of $E$ such that every subfamily has nonempty intersection, then $\bigcap_{F \in \mathcal{F}} F \neq \varnothing$.

(iii) $E$ is complete and *totally bounded i.e.* can be covered by finitely many open balls of radius $\varepsilon$, for any $\varepsilon > 0$.

(iv) $E$ has the *Bolzano-Weierstrass* property, *i.e.* is sequentially compact.