MATH 113 - ANALYSIS
SPRING 2015
IN-CLASS MIDTERM

DURATION: 3 HOURS

This exam consists of three independent problems. You may treat them in the order of your choosing.

If you were not able to solve a question but wish to use the result to solve another one, you are welcome to do so, as long as you indicate it explicitly.

Notation: if (E, d) is a metric space, $x \in E$ and $r > 0$, we denote by $B_E(x, r)$ the *open* ball centered at x with radius r, that is,

$$B_E(x, r) = \{y \in E \,,\ d(x, y) < r\}.$$

Reminder: a useful consequence of the Baire Category Theorem is the following.

Proposition. *If E is a Baire space and $\{F_n\}_{n\geq 1}$ is a sequence of closed subsets such that $\bigcup_{n\geq 1} F_n = E$, then $\bigcup_{n\geq 1} \overset{\circ}{F_n}$ is a dense open subset of E.*

Problem 1

1. Is $c_0(\mathbb{N}) = \big\{\{u_n\} \in \mathbb{R}^{\mathbb{N}} \,,\ \lim_{n\to\infty} u_n = 0\big\}$ complete for the norm $\|\{u_n\}\|_\infty = \sup_{n\in\mathbb{N}} |u_n|$?

2. Is $C([0, 1], \mathbb{R})$ complete for the norm $\|f\|_1 = \displaystyle\int_0^1 |f(x)|\, dx$?

Problem 2

Let E and F be Banach spaces. We denote by $\mathbb{B}$ the closed ball of radius 1 in E, that is, $\mathbb{B} = \overline{B_E(0,1)}$. A bounded operator $T \in \mathcal{L}(E,F)$ is said *compact* if $\overline{T(\mathbb{B})}$ is compact.

1. Characterize the Banach spaces E such that the identity map Id_E is compact.

2. Assume that $T \in \mathcal{L}(E,F)$ has finite-dimensional range. Prove that T is compact.

3. Let $T \in \mathcal{L}(E,F)$ be compact and assume that the range $r(T)$ of T is closed in F.

a. Show the existence of $\rho > 0$ such that $B_{r(T)}(0,\rho) \subset T(\mathbb{B})$.

b. Prove that $r(T)$ is finite-dimensional.

4. *Integral operators with continuous kernels are compact.*

Let $E = (C([0,1]), \|\cdot\|_\infty)$. For $\kappa \in C([0,1]^2)$, we define a linear map $T : E \longrightarrow E$ by

$$T(f)(x) = \int_0^1 \kappa(x,y) f(y)\, dy.$$

a. Prove that T is continuous.

b. Prove that T is compact.

Problem 3

1. Let (E,d) and (F,δ) be metric spaces. Assume E complete and consider a sequence $\{f_n\}_{n\geq 1}$ of continuous maps from E to F that converges pointwise to $f : E \longrightarrow F$.

a. Consider, for $n \geq 1$ and $\varepsilon > 0$, the set

$$F_{n,\varepsilon} = \{x \in E \quad \text{s. t.} \quad \forall p \geq n\ ,\ \delta(f_n(x), f_p(x)) \leq \varepsilon\}\,.$$

Show that $\Omega_\varepsilon = \bigcup_{n\geq 1} \overset{\circ}{F}_{n,\varepsilon}$ is a dense open subset of E.

b. Show that every point $x_0 \in \Omega_\varepsilon$ has a neighborhood $\mathcal{N}$ such that

$$\forall x \in \mathcal{N}\ ,\ \delta(f(x_0), f(x)) \leq 3\varepsilon.$$

c. Prove that f is continuous at every point of $\Omega = \bigcap_{n\geq 1} \Omega_{\frac{1}{n}}$ and that $\overline{\Omega} = E$.

2. *Application:* let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function. Show that its derivative f' is continuous on a dense subset of $\mathbb{R}$.