Mathematics 111

Spring 2011

Homework 4

1. (A theorem not proven in class). Let R be a ring with identity, M a right R-module and N a free left R-module with basis $\{e_i\}_{i\in I}$. Show that every element in $M \otimes_R N$ can be written uniquely as a finite sum $\sum_{i\in I} m_i \otimes e_i$, $m_i \in M$. Here uniquely means that if $\sum_i m_{i_k} \otimes e_{i_k} = \sum_i m'_{i_k} \otimes e_{i_k}$, then $m_{i_k} = m'_{i_k}$ for all i_k.

   Hint: Use the fact that tensor product commutes with the coproduct to reduce to the one-dimensional case. Then prove that as abelian groups $M \otimes_R Re_i \cong M$ by exhibiting inverse homomorphisms between them.

2. Show that, as a corollary to the above problem, if $R \subset S$ are rings with $1_R = 1_S$, then for any free left R-module N with basis $\{e_i\}_{i\in I}$, $S \otimes_R N$ is a free left S-module with basis $\{1_S \otimes e_i\}_{i\in I}$.

3. (problem 25, D&F, 10.4) Let $R \subset S$ be commutative rings with $1_R = 1_S$. Show that as S-algebras, $S \otimes_R R[x] \cong S[x]$, where x is an indeterminate over S. Be sure to pay attention to justifying that your isomorphism is S-linear.

4. (problem 26, D&F, 10.4) Let $R \subset S$ be commutative rings with $1_R = 1_S$, and let I be an ideal in the polynomial ring $R[x_1, \ldots, x_n]$. Show that as S-algebras,
   $S \otimes_R (R[x_1, \ldots, x_n]/I) \cong S[x_1, \ldots, x_n]/IS[x_1, \ldots, x_n]$.

5. Let $L = \mathbb{Q}(\zeta_5)$ be the fifth cyclotomic field and let $K = \mathbb{Q}(\sqrt[5]{2})$. Using problem 4, determine $L \otimes_{\mathbb{Q}} K$ as an L-algebra. It is a vector space over L, and its dimension can be computed in two ways one of which uses field theory. Are your answers consistent?

6. Using problem 4, compute $\mathbb{Q}(\sqrt{2}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt[4]{2})$ as a $\mathbb{Q}$-algebra in two ways (thinking of the tensor product as an extension of scalars in two ways). Are your summands all isomorphic as $\mathbb{Q}$-algebras? Why or why not?