Mathematics 111

Spring 2009

Homework 6

1. Let f be a monic polynomial of degree $n \geq 1$, and let $C(f)$ be the companion matrix.

   (a) Although we have shown in class that the minimal polynomial of $C(f)$ is f, give a proof (independent of this fact) that the characteristic polynomial of $C(f)$ is f.

   (b) Let T be a linear operator on a finite dimensional vector space V over a field k, and let $q_1 \mid q_2 \mid \cdots \mid q_s$ be the invariant factors associated to V as a torsion $k[x]$-module. Show that $q_1 q_2 \cdots q_s = c_T$, where c_T is the characteristic polynomial of T. Show that the minimal polynomial of T divides the characteristic polynomial. Note that this proves the Cayley-Hamilton theorem. *Hint:* You may use without proof (though you should think about it) that the determinant of a matrix of the form $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ with A and C square matrices is the product $\det(A)\det(C)$.

2. Find all rational and Jordan canonical forms of a matrix in $M_5(\mathbb{C})$ having minimal polynomial $x^2(x-1)$. Be sure to give the corresponding invariants and the characteristic polynomials.

3. Show that any linear operator T on a finite dimensional vector space (over a field of characteristic not equal to 2) which satisfies $T^2 = I$ is diagonalizable. Give all possible Jordan forms for 4×4 matrices A with $A^2 = I$.

4. Consider a matrix of the form $A = \begin{pmatrix} \lambda & \mu & 0 & \dots & 0 \\ 0 & \lambda & \mu & \dots & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & \dots & \lambda & \mu \\ 0 & 0 & \dots & 0 & \lambda \end{pmatrix}$, i.e., with diagonal λ and μ on the superdiagonal. Find the Jordan canonical form(s) of A. The answer should depend slightly on μ. What does your answer say about the form of Jordan blocks as introduced in the text in comparison with the way we defined them?

5. Let T be a linear operator on a finite dimensional vector space V. Let $\lambda_1, \dots, \lambda_k$ be distinct eigenvalues of T and let $v_1, \dots, v_k$ be corresponding (non-zero) eigenvectors. Show that $\{v_1, \dots, v_k\}$ is linearly independent. *Hint:* Apply $T - \lambda_k I$ to a dependence relation and use induction.