5. ARE REAL DICE FAIR?

According to Galileo (section 1), when a die is rolled it is equally likely to show any of its 6 faces. Galileo was thinking of an ideal die which is perfectly symmetric. This is like ignoring friction in the study of physics: the results are only a first approximation. What does Galileo's calculation say about real dice?

- For real dice, the 216 possible ways three dice can land are close to being equally likely.
- If these ways were equally likely, the chance of rolling a total of 9 spots would be exactly 25 in 216.
- So for real dice, the chance of rolling a total of 9 spots is just about 25 in 216.

For loaded dice, the calculations would be badly off. But ordinary dice, coins, and the like are very close to fair—in the sense that all the outcomes are equally likely. Of course, you have to put some effort into shaking the dice or flipping the coins. And the games of chance based on these fair mechanisms may be quite unfair (chapter 17).

In a similar way, if you are told that a ticket is drawn at random, you should assume that each ticket in the box is equally likely to be drawn. If the tickets are close to the same size, shape, and texture, and the box is well shaken, this is quite a reasonable approximation.

6. REVIEW EXERCISES

Review exercises may cover material from previous chapters.

When a die is rolled, each of the 6 faces is equally likely to come up. A deck of cards has 4 suits (clubs, diamonds, hearts, spades) with 13 cards in each suit—2, 3, . . . , 10, jack, queen, king, ace. See pp. 222 and 226.

1. A pair of dice are thrown.
   (a) Find the chance that both dice show 3 spots.
   (b) Find the chance that both dice show the same number of spots.
2. In the game of Monopoly, a player rolls two dice, counts the total number of spots, and moves that many squares. Find the chance that the player moves 11 squares (no more and no less).
3. True or false, and explain:
   (a) If a die is rolled three times, the chance of getting at least one ace is $1/6 + 1/6 + 1/6 = 1/2$.
   (b) If a coin is tossed twice, the chance of getting at least one head is 100%.
4. Two cards will be dealt off the top of a well-shuffled deck. You have a choice:

(i) to win \$1 if at least one of the two cards is a queen.
(ii) to win \$1 if the first is a queen.

Which option is better? Or are they equivalent? Explain.

5. The chance of A is 1/3; the chance of B is 1/10. True or false, and explain:
   (a) If A and B are independent, they must also be mutually exclusive.
   (b) If A and B are mutually exclusive, they cannot be independent.

6. One event has chance 1/2, another has chance 1/3. Fill in the blanks, using one phrase from each pair below, to make up two true sentences. Write out both sentences.

   "If you want to find the chance that ___(i)___ will happen, check to see if they are ___(ii)___. If so, you can ___(iii)___ the chances."

   (i) at least one of the two events, both events
   (ii) independent, mutually exclusive
   (iii) add, multiply

7. Four draws are going to be made at random with replacement from the box $\boxed{1}\boxed{2}\boxed{2}\boxed{3}\boxed{3}$. Find the chance that $\boxed{2}$ is drawn at least once.

8. Repeat exercise 7, if the draws are made at random without replacement.

9. One ticket will be drawn at random from each of the two boxes shown below:

   (A) $\boxed{1}\boxed{2}\boxed{3}$ (B) $\boxed{1}\boxed{2}\boxed{3}\boxed{4}$

   Find the chance that:
   (a) The number drawn from A is larger than the one from B.
   (b) The number drawn from A equals the one from B.
   (c) The number drawn from A is smaller than the one from B.

10. There are two options:
    (i) A die will be rolled 60 times. Each time it shows an ace or a six, you win \$1; on the other rolls, you win nothing.
    (ii) Sixty draws will be made at random with replacement from the box $\boxed{1}\boxed{1}\boxed{1}\boxed{0}\boxed{0}\boxed{0}$. On each draw, you will be paid the amount shown on the ticket, in dollars.

    Which option is better? or are they the same? Explain briefly.

11. Three cards are dealt from a well-shuffled deck.
    (a) Find the chance that all of the cards are diamonds.
    (b) Find the chance that none of the cards are diamonds.
    (c) Find the chance that the cards are not all diamonds.

12. A coin is tossed 10 times. True or false, and explain:
    (a) The chance of getting 10 heads in a row is 1/1,024.
    (b) Given that the first 9 tosses were heads, the chance of getting 10 heads in a row is 1/2.

Exercises 13 and 14 are more difficult.

13. A box contains 2 red marbles and 98 blue ones. Draws are made at random with replacement. In _____ draws from the box, there is better than a 50% chance for a red marble to appear at least once. Fill in the blank with the smallest number that makes the statement true. (You will need a calculator.)

14. In Lotto 6-53, there is a box with 53 balls, numbered from 1 to 53. Six balls are drawn at random without replacement from the box. You win the grand prize if the numbers on your lottery ticket are the same as the numbers on the six balls; order does not matter.

    Person A bought two tickets, with the following numbers:

Ticket #1	5	12	21	30	42	51
Ticket #2	5	12	23	30	42	49

    Person B bought two tickets, with the following numbers:

Ticket #1	7	11	25	28	34	50
Ticket #2	9	14	20	22	37	45

    Which person has the better chance of winning? Or are their chances the same? Explain briefly.

7. SUMMARY

1. When figuring chances, one helpful strategy is to write down a complete list of all the possible ways that the chance process can turn out. If this is too hard, at least write down a few typical ways, and count how many ways there are in total.

2. The chance that at least one of two things will happen equals the sum of the individual chances, provided the things are mutually exclusive. Otherwise, adding the chances will give the wrong answer—double counting.

3. If you are having trouble working out the chance of an event, try to figure out the chance of its opposite; then subtract from 100%.

number comes up, the gambler gets the dollar back, together with winnings of \$17. If neither number comes up, he loses the dollar. So a split pays 17 to 1, and there are 2 chances in 38 to win. The gambler's net gain in the 25 plays is like the sum of 25 draws made from one of the following boxes. Which one, and why?

(i) | 0 | 00 | 36 tickets numbered | 1 | through | 36 |

(ii) | \$17 | \$17 | 34 tickets | −\$1 |

(iii) | \$17 | \$17 | 36 tickets | −\$1 |

3. In one version of chuck-a-luck, 3 dice are rolled out of a cage. You can bet that all 3 show six. The house pays 36 to 1, and the bettor has 1 chance in 216 to win. Suppose you make this bet 10 times, staking \$1 each time. Your net gain is like the sum of _____ draws made at random with replacement from the box _____. Fill in the blanks.

The answers to these exercises are on p. A72.

5. REVIEW EXERCISES

1. A box contains 10,000 tickets: 4,000 | 0 |'s and 6,000 | 1 |'s. And 10,000 draws will be made at random with replacement from this box. Which of the following best describes the situation, and why?
   (i) The number of 1's will be 6,000 exactly.
   (ii) The number of 1's is very likely to equal 6,000, but there is also some small chance that it will not be equal to 6,000.
   (iii) The number of 1's is likely to be different from 6,000, but the difference is likely to be small compared to 10,000.

2. Repeat exercise 1 for 10,000 draws made at random without replacement from the box.

3. A gambler loses ten times running at roulette. He decides to continue playing because he is due for a win, by the law of averages. A bystander advises him to quit, on the grounds that his luck is cold. Who is right? Or are both of them wrong?

4. (a) A die will be rolled some number of times, and you win \$1 if it shows an ace (| • |) more than 20% of the time. Which is better: 60 rolls, or 600 rolls? Explain.
   (b) As in (a), but you win the dollar if the percentage of aces is more than 15%.
   (c) As in (a), but you win the dollar if the percentage of aces is between 15% and 20%.
   (d) As in (a), but you win the dollar if the percentage of aces is exactly $16\frac{2}{3}$%.

5. True or false: if a coin is tossed 100 times, it is not likely that the number of heads will be exactly 50, but it is likely that the percentage of heads will be exactly 50%. Explain.

6. (a) A coin is tossed 10,000 times. What is the chance that the number of heads will be in the range 4,850 to 5,150?
   (b) A coin is tossed 1,000,000 times. What is the chance that the number of heads will be in the range 498,500 to 501,500?

7. Fifty draws are made at random with replacement from the box $\boxed{0}\boxed{0}\boxed{1}\boxed{1}\boxed{1}$; there are 33 $\boxed{1}$'s among the draws. The expected number of $\boxed{1}$'s is _____, the observed number is _____, the chance error is _____, and the SE is _____.

8. A computer program is written to do the following job. There is a box with ten blank tickets. You tell the program what numbers to write on the tickets, and how many draws to make. Then, the computer will draw that many tickets at random with replacement from the box, add them up, and print out the sum—but not the draws. This program does not know anything about coin tossing. Still, you can use it to simulate the number of heads in 1,000 tosses of a coin. How?

9. A die is rolled 100 times. Someone figures the expected number of aces as $100 \times 1/6 = 16.67$, and the SE as $\sqrt{100} \times \sqrt{1/6 \times 5/6} \approx 3.73$. (An ace is $\boxed{\cdot}$.) Is this right? Answer yes or no, and explain.

The answers to these exercises are on p. A75.

6. REVIEW EXERCISES

1. One hundred draws will be made at random with replacement from the box $\boxed{1}\boxed{6}\boxed{7}\boxed{9}\boxed{9}\boxed{10}$.
   (a) How small can the sum of the draws be? How large?
   (b) The sum is between 650 and 750 with a chance of about
   1% 10% 50% 90% 99%
   Explain.

2. A gambler plays roulette 100 times, betting a dollar on a column each time. The bet pays 2 to 1, and there are 12 chances in 38 to win. Fill in the blanks; show work.
   (a) In 100 plays, the gambler's net gain will be around \$_____, give or take \$_____ or so.
   (b) In 100 plays, the gambler should win _____ times, give or take _____ or so.
   (c) How does the column bet compare with betting on a single number at Keno (example 1 on p. 289)?

3. Match the lists with the SDs. Explain your reasoning

(a) 1, −2, −2	(i) $\sqrt{1/3 \times 2/3}$
(b) 15, 15, 16	(ii) $2 \times \sqrt{1/3 \times 2/3}$
(c) −1, −1, −1, 1	(iii) $3 \times \sqrt{1/3 \times 2/3}$
(d) 0, 0, 0, 1	(iv) $\sqrt{1/4 \times 3/4}$
(e) 0, 0, 2	(v) $2 \times \sqrt{1/4 \times 3/4}$

4. A large group of people get together. Each one rolls a die 180 times, and counts the number of [•]'s. About what percentage of these people should get counts in the range 15 to 45?

5. A die will be thrown some number of times, and the object is to guess the total number of spots. There is a one-dollar penalty for each spot that the guess is off. For instance, if you guess 200 and the total is 215, you lose $15. Which do you prefer: 50 throws, or 100? Explain.

6. One hundred draws are made at random with replacement from the box [1 1 2 3]. The draws come out as follows: 45 [1]'s, 23 [2]'s, 32 [3]'s. For each number below, find the phrase which describes it.

Number	*Phrase*
12	observed value for the sum of the draws
45	observed value for the number of 3's
187	observed value for the number of 1's
25	expected value for the sum of the draws
50	expected value for the number of 3's
175	expected value for the number of 1's
5	chance error in the sum of the draws
32	standard error for the number of 1's

7. One hundred draws are made at random with replacement from the box [1 2 3 4 5 6].
   (a) If the sum of the draws is 321, what is their average?
   (b) If the average of the draws is 3.78, what is the sum?
   (c) Estimate the chance that the average of the draws is between 3 and 4.

8. A coin is tossed 100 times.
   (a) The difference "number of heads – number of tails" is like the sum of 100 draws from one of the following boxes. Which one, and why?
   (i) [heads tails]
   (ii) [−1 1]
   (iii) [−1 0]
   (iv) [0 1]
   (v) [−1 0 1]
   (b) Find the expected value and standard error for the difference.

9. A gambler plays roulette 1,000 times. There are two possibilities:
   (i) Betting $1 on a column each time.
   (ii) Betting $1 on a number each time.
   A column pays 2 to 1, and there are 12 chances in 38 to win; a number pays 35 to 1, and there is 1 chance in 38 to win. True or false and explain:

(a) The chance of coming out ahead is the same with (i) and (ii).
(b) The chance of winning more than \$100 is bigger with (ii).
(c) The chance of losing more than \$100 is bigger with (ii).

10. A box contains numbered tickets. Draws are made at random with replacement from the box. Below are three statements about this particular box; (i) and (ii) are true. Is (iii) true or false? Explain.
    (i) For a certain number of draws, the expected value for the sum of the draws equals 400.
    (ii) For that same number of draws, there is about a 75% chance that the sum will be between 350 and 450.
    (iii) For twice that number of draws, there is about a 75% chance that the sum will be between 700 and 900.

11. One hundred draws are made at random with replacement from the box $\boxed{-2}\boxed{-1}\boxed{0}\boxed{1}\boxed{3}$. The sum of the positive numbers will be around _____, give or take _____ or so.

12. One hundred draws are made at random with replacement from the box $\boxed{1}\boxed{2}\boxed{3}\boxed{4}\boxed{5}\boxed{6}\boxed{7}$.
    (a) The sum of the draws is 431. The expected value for the sum of the draws is _____, the observed value is _____, the chance error is _____, and the standard error is _____.
    (b) The sum of the draws is 386. The expected value for the sum of the draws is _____, the observed value is _____, the chance error is _____, and the standard error is _____.
    (c) The sum of the draws is 417. The expected value for the sum of the draws is _____, the observed value is _____, the chance error is _____, and the standard error is _____.

13. A letter is drawn 1,000 times, at random, from the word A R A B I A. There are two offers.
    (A) You win a dollar if the number of A's among the draws is 10 or more above the expected number.
    (B) You win a dollar if the number of B's among the draws is 10 or more above the expected number.

    Choose one option and explain.
    (i) A gives a better chance of winning than B.
    (ii) A and B give the same chance of winning.
    (iii) B gives better chance of winning than A.
    (iv) There is not enough information to decide.

14. In roulette, once in a while, someone will bet \$1 on red; and, at the same time, someone else will bet \$1 on black (p. 282). Suppose this pair of bets is made 100 times in the course of an evening.
    (a) The house will make money on _____ of the 100 pairs of bets, give or take _____ or so.

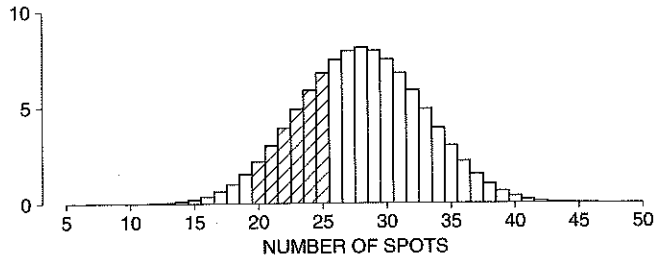
from the box got larger and larger. Then the probability histogram for the sum got smoother and smoother, and in the limit became the normal curve. Empirical histograms are one thing; probability histograms quite another.

In part II of the book, the normal curve was used for data. In some cases, this can be justified by a mathematical argument which uses the two types of convergence discussed in this chapter. When the number of repetitions is large, the empirical histogram will be close to the probability histogram. When the number of draws is large, the probability histogram for the sum will be close to the normal curve. Consequently, when the number of repetitions and the number of draws are both large, the empirical histogram for the sums will be close to the curve.[11] This is all a matter of pure logic: a mathematician can prove every step.

But there is still something missing. It has to be shown that the process generating the data is like drawing numbers from a box and taking the sum. This sort of argument will be discussed in part VII. More than mathematics is involved—there will be questions of fact to settle.

7. REVIEW EXERCISES

1. The figure below shows the probability histogram for the total number of spots when a die is rolled eight times. The shaded area represents the chance that the total will be between _____ and _____ (inclusive).



2. Four hundred draws will be made at random with replacement from the box $\boxed{1}\boxed{3}\boxed{5}\boxed{7}$.

   (a) Estimate the chance that the sum of the draws will be more than 1,500.
   (b) Estimate the chance that there will be fewer than 90 $\boxed{3}$'s.

3. Ten draws are going to be made at random with replacement from the box $\boxed{0}\boxed{1}\boxed{2}\boxed{3}$. The chance that the sum will be in the interval from 10 to 20 inclusive equals the area under _____ between _____ and _____. Fill in the blanks. For the first one, your options are: the normal curve, the probability histogram for the sum. Explain your answers.

4. A coin is tossed 25 times. Estimate the chance of getting 12 heads and 13 tails.