Math 105: Introduction to Modular Forms
Homework 2: Due Wednesday, May 7

1. (a) Prove that $\{I\} \cup \{T^{-j}S\}_{j=0}^{p-1}$ is a complete set of left coset representatives for $\Gamma_0(p)$ in Γ.

   (b) Draw a fundamental domain for $\Gamma_0(2)$.

2. Let $f \in M_k(\Gamma_0(N), \chi)$ and set $W_N := \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$. The map $f \mapsto f|_k W_N$ is called the Fricke involution.

   (a) Prove that $f|_k W_N \in M_k(\Gamma_0(N), \overline{\chi})$, and that the square of the Fricke involution is the map $(-1)^k$ on $M_k(\Gamma_0(N), \chi)$. Show that the Fricke involution is an isomorphism of vector spaces from $M_k(\Gamma_0(N), \chi)$ to $M_k(\Gamma_0(N), \overline{\chi})$.

   (b) If $\chi = \overline{\chi}$ (that is if $\chi(d) = \pm 1$), then show that
$$M_k(\Gamma_0(N), \chi) = M_k^+(\Gamma_0(N), \chi) \oplus M_k^-(\Gamma_0(N), \chi),$$
   where $M_k^{\pm}(\Gamma_0(N), \chi) = \{f \in M_k(\Gamma_0(N), \chi) : f|_k W_N = \pm i^{-k} f\}$.

3. (a) For $\delta \in \mathbb{N}$, show that if $\gamma \in \Gamma$ then $\begin{pmatrix} \delta & 0 \\ 0 & 1 \end{pmatrix} \gamma = \gamma' \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ for some $\gamma' \in \Gamma$ and integers A, B and D, with A and D positive.

   (b) For $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{Q})$, define
$$\eta(z)|_{\frac{1}{2}}\alpha := (\det \alpha)^{\frac{1}{4}} (cz + d)^{-\frac{1}{2}} \eta(\alpha z),$$
   where we take the branch of the square root having nonnegative real part. Using the fact that for $\gamma \in \Gamma$, $\eta(z)|_{\frac{1}{2}}\gamma = \epsilon\eta(z)$ for some root of unity ϵ, prove that the order of vanishing of an eta-quotient $f(z) = \prod_{\delta | N} \eta^{r_\delta}(\delta z)$ at a cusp s is
$$\mathrm{ord}_s f = \frac{N}{24(c^2, N)} \sum_{\delta | N} \frac{(c, \delta)^2}{\delta} r_\delta.$$

4. (a) Show that $g(z) = \eta^3(z)\eta^3(7z) \in S_3\left(\Gamma_0(7), \left(\frac{\cdot}{7}\right)\right)$.

   (b) Prove that if f is a nonzero element of $S_3\left(\Gamma_0(7), \left(\frac{\cdot}{7}\right)\right)$, then it is a constant multiple of $g(z)$.