Math 103: Measure Theory and Complex Analysis
Fall 2018

09/14/18

Lecture 2

Aim: Create a good theory of measure and measurable sets.

We recall:
Definition 1 (σ algebra) Let X be a set, a collection $\mathcal{M} \subset \mathcal{P}(X)$ of subsets of X is called a σ **algebra** if

a) $X \in \mathcal{M}$.

b) $A \in \mathcal{M} \Rightarrow A^c = X \backslash A \in \mathcal{M}$ ($\mathcal{M}$ is closed under complements).

c) $(A_k)_{k=1,\dots,\infty} \subset \mathcal{M} \Rightarrow \bigcup_{k=1}^{\infty} A_k \in \mathcal{M}$ ($\mathcal{M}$ is closed under countable unions).

In this case $(X, \mathcal{M})$ is called a **measurable space**.

Lemma 2 If $\mathcal{M}$ is a σ algebra then

a) $\emptyset \in \mathcal{M}$.

b) $A, B \in \mathcal{M} \Rightarrow A \backslash B \in \mathcal{M}$.

c) $(A_k)_{k=1,\dots,n} \subset \mathcal{M} \Rightarrow \bigcup_{k=1}^{n} A_k \in \mathcal{M}$ ($\mathcal{M}$ is closed under finite unions).

d) $(A_k)_{k\in\mathbb{N}} \subset \mathcal{M} \Rightarrow \bigcap_{k=1}^{\infty} A_k \in \mathcal{M}$ ($\mathcal{M}$ is closed under countable intersections).

proof

Examples
1.) $\mathcal{M} = \{\emptyset, X\}$
2.) $\mathcal{M} = \mathcal{P}(X)$
3.) Assume that X is uncountable and set

$$\mathcal{M} = \{A \subset X \mid X \text{ is countable or } X^c \text{ is countable}\} \; \textbf{(HW 1)}$$

Math 103: Measure Theory and Complex Analysis
Fall 2018

09/14/18

We can generate σ algebras from interesting subsets of $\mathcal{P}(X)$:

Proposition 3 Let X be a set and $\mathcal{S} \subset \mathcal{P}(X)$ be a collection of subsets of X. Then there is a smallest σ algebra $\mathcal{M}$ such that $\mathcal{S} \subset \mathcal{M}$.
We call $\mathcal{M} = \langle \mathcal{S} \rangle$ the σ **algebra generated by $\mathcal{S}$.**

proof Let $\mathcal{C}$ be the set of all σ algebras $\mathcal{M}'$, such that $\mathcal{S} \subset \mathcal{M}'$. Note that $\mathcal{C} \neq \emptyset$ as $\mathcal{P}(X) \in \mathcal{C}$. We check that

$$\mathcal{M} = \bigcap_{\mathcal{M}' \in \mathcal{C}} \mathcal{M}'$$

is a σ algebra:

Clearly it is the smallest such. $\square$

Generating σ algebras from topology

Aim: We can generate σ algebras from the topology of a set. This connects continuity and measurability.

Reminder: Topology

Definition 4 Let X be a set. A **topology** on X is a collection $\mathcal{T} \subset \mathcal{P}(X)$ of subsets of X, such that

a) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$.

b) $A, B \in \mathcal{T} \Rightarrow A \cap B \in \mathcal{T}$ ($\mathcal{T}$ is closed under intersection).

c) $(A_k)_{k\in K} \subset \mathcal{T} \Rightarrow \bigcup_{k\in K} A_k \in \mathcal{T}$ ($\mathcal{T}$ is closed under **any** union).

In this case the elements of $\mathcal{T}$ the **open subsets** of X and $(X, \mathcal{T})$ is called a **topological space**.

Examples $\mathcal{T} = \{\emptyset, X\}$ or $\mathcal{T}' = \mathcal{P}(X)$.

Remark b) implies that $\mathcal{T}$ is closed under finite intersections.

Math 103: Measure Theory and Complex Analysis
Fall 2018

09/14/18

Definition 5 Let $(X, \mathcal{T})$ and $(X', \mathcal{T}')$ be topological spaces. A function $f : X \to X'$ is **continuous** if

$$f^{-1}(A) \in \mathcal{T} \quad \text{for all} \quad A \in \mathcal{T}'.$$

Remark Continuity depends on the topologies.

Picture:

Similarly we can define a measurable function.

Definition 6 Let $(X, \mathcal{M})$ and $(X', \mathcal{M}')$ be measurable spaces. A function $f : X \to X'$ is **measurable** if

$$f^{-1}(A) \in \mathcal{M} \quad \text{for all} \quad A \in \mathcal{M}'.$$

We can now generate σ algebras from topologies and define measurable functions for these σ algebras:

Definition 7 (Borel sets) If $(X, \mathcal{T})$ is a topological space, then the **Borel sets** are the elements of the σ algebra

$$\mathcal{B}(X) = \langle \mathcal{T} \rangle, \quad \text{see } \textbf{Proposition 3}.$$

Remark Although $\mathcal{T}$ determines the σ algebra it is usually omitted from the definition.

Convention If $(X, \mathcal{M})$ is a measurable space and $(Y, \mathcal{T})$ a topological space then we say that

$$f : X \to Y \quad \text{is measurable}$$

if and only it is measurable with respect to $\mathcal{M}$ and $\mathcal{B}(Y)$.

Proposition 8 In the situation above, f is measurable if and only if

$$f^{-1}(A) \in \mathcal{M} \quad \text{for all} \quad A \in \mathcal{T}.$$

This means we can simplify our task when checking if a function is measurable. Before we prove the proposition, we prove a short lemma:

Lemma 9 (induced σ algebra) Let $f : X \to Y$ be a function and $(X, \mathcal{M})$ be measurable space. Then

$$\mathcal{N} = \{B \in Y \mid f^{-1}(B) \in \mathcal{M}\}$$

is a σ algebra in Y, such that f is measurable.

---

proof We have to show that $\mathcal{N}$ is a σ algebra. We check the conditions:

By a) - c) we know that $\mathcal{N}$ is a σ algebra. By definition f is measurable for with respect to $\mathcal{M}$ and $\mathcal{N}$. $\square$

We note that this σ algebra is induced by f. In a similar fashion, we can also induce a topology on Y via f, such that f is continuous. If $\mathcal{T}$ is a topology of X then

$$\mathcal{T}' = \{B \in Y \mid f^{-1}(B) \in \mathcal{T}\}$$

is a topology in Y, such that f is continuous.

We now prove the proposition:

proof " $\Rightarrow$ " By definition the generating set of the σ algebra is $\mathcal{T}$ and by the definition of $\langle \mathcal{T} \rangle$ we have that $\mathcal{T} \subset \langle \mathcal{T} \rangle = \mathcal{B}(Y)$. As f is measurable this implies one direction of the proposition.

" $\Leftarrow$ "

$\square$

Math 103: Measure Theory and Complex Analysis
Fall 2018

09/14/18

Definition 10 (Borel map) Let X, Y be topological spaces. A map $f : X \to Y$ is called a **Borel map** or also **Borel** if $f^{-1}(B)$ is in $\mathcal{B}(X)$ for all $B \in Y$ open.

Part 1.) - 3.) of the following remark follow from **Proposition 8**. Part 4.) follows directly from the definition of measurable functions.

Remark 11

1.) f Borel $\Leftrightarrow$ f measurable for $\mathcal{B}(X)$ and $\mathcal{B}(Y)$.

2.) f continuous $\Rightarrow$ f Borel.

3.) If $f : X \to Y$ is Borel and $g : (Z, \mathcal{M}) \to X$ measurable. Then $f \circ g : Z \to Y$ is measurable.

4.) The composition of measurable functions is measurable.

We have seen it the proposition that it is sufficient to check the elements of the topology to see if f is measurable with respect to the generated σ algebra.

Question Is it sufficient to check only the basis of a topology to see if a function is measurable?
Answer This is sufficient, but only if the topology has a countable basis.

Definition (basis) Let $(X, \mathcal{T})$ be a topological space. Then $\beta \subset \mathcal{T}$ is a basis for the topology $\mathcal{T}$ if

$$\text{for all} \quad A \in \mathcal{T} \quad \text{we have that} \quad A = \bigcup_{i \in I} A_i \quad \text{where} \quad (A_i)_{i \in I} \subset \beta.$$

This means that every element in $\mathcal{T}$ is a union of elements of β.

Theorem 12 (basis = neighbourhood basis) β is a basis for the topology $\mathcal{T}$ iff

$$\text{for all } A \in \mathcal{T} \text{ and for all } x \in A \ \exists \, \mathcal{U}(x) = \mathcal{U} \in \beta, \quad \text{such that} \quad x \in \mathcal{U} \subset A.$$

Picture

Example A basis for $\mathbb{R}$ is given by the set of open intervals (a, b) in $\mathbb{R}$.

Math 103: Measure Theory and Complex Analysis
Fall 2018

09/14/18

Definition (second countable) A topological space $(X, \mathcal{T})$ is called **second countable** if there is a countable basis for its topology.

Example A countable basis for the usual topology of the real line $\mathbb{R}$ is given by the open intervals with rational endpoints.

Proposition 13 If (X, d) is a metric space with a countable dense subset, the topology induced by the metric is second countable.

proof We know that
Picture
1.) 3.)

1.) the basis β_d of the topology $\mathcal{T}_d$ induced by the metric d is the collection of open balls in (X, d): $\beta_d = \{B_r(x) \mid r \in \mathbb{R}^+, x \in X\}$

2.) there is a countable dense subset $D = (x_n)_{n\in\mathbb{N}}$ in X.

3.) by **Theorem 12**, as β_d is a basis, we know that for all $A \in \mathcal{T}_d$ and $x \in A$ there is $B_r(x') \subset \beta_d$, such that $x \in B_r(x') \subset A$.

$\square$

Corollary For $n \geq 1$, $\mathbb{R}^n$ is second countable in the usual topology.

Lemma 14 Suppose β is a countable basis for the topology $\mathcal{T}'$ of Y. Then

$$f : (X, \mathcal{M}) \to (Y, \mathcal{B}(Y)) \text{ measurable } \Leftrightarrow f^{-1}(B) \in \mathcal{M} \text{ for all } B \in \beta.$$

proof By **Proposition 8** we know that it is sufficient to show that

$$f^{-1}(B) \in \mathcal{M} \text{ for all } B \in \beta \Leftrightarrow f^{-1}(V) \in \mathcal{M} \text{ for all } V \in \mathcal{T}'$$

"$\Rightarrow$"

"$\Leftarrow$" The other direction is trivial. $\square$