Math 103: Measure Theory and Complex Analysis
Fall 2018

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Lecture 25

Chapter 5 - Local behavior

Theorem 1 Suppose Ω is a convex region and $f \in \mathcal{H}(\Omega)$ has finitely many zeroes $Z(f) = \{a_1, \ldots, a_n\}$ in Ω with each zero repeated according to its multiplicity. Let γ be a closed path in Ω such that $\gamma^* \cap Z(f) = \varnothing$. Then

$$\sum_{j=1}^{n} \text{Ind}_\gamma(a_j) = \frac{1}{2i\pi} \int_\gamma \frac{f'(w)}{f(w)} dw$$

Picture

proof of Theorem 1 By induction, $\exists h \in \mathcal{H}(\Omega)$ such that $f(z) = (z - a_1) \cdots (z - a_n) h(z)$ and $h(z) \neq 0$ for $z \in \Omega$:

Furthermore, by induction:

$$\frac{f'(z)}{f(z)} = \frac{1}{z - a_1} + \cdots + \frac{1}{z - a_n} + \frac{h'(z)}{h(z)}$$

proof

Since $\frac{h'}{h} \in \mathcal{H}(\Omega)$ as $h(z) \neq 0$ and Ω is convex, $\int_\gamma \frac{h'(z)}{h(z)} dz = 0$ and our statement follows from the definitions. $\square$

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Corollary 2 If $\gamma = \partial D$ is a positively oriented circle, then $N_f = \frac{1}{2i\pi}\int_\gamma \frac{f'(z)}{f(z)}dz$ is the number of zeroes of f in D.

Picture

Let Ω be a region, $f \in \mathcal{H}(\Omega)$ and $\overline{D_r(a)} \subset \Omega$. Then $\exists r' > r$ such that $\overline{D_{r'}(a)} \subset \Omega$.

Note that $D_{r'}(a)$ is convex. If f is a non-constant function on $D_{r'}(a)$, then $\boxed{g_w = f - w}$ has finitely many zeros in $D_{r'}(a)$, $\{a_{1(w)}, \ldots, a_{n(w)}\} \subset D_{r'}(a)$.

$D_{r'}(a)$ is convex. Let $\gamma = \partial D_r(a)$ and $\Gamma = f \circ \gamma$.

$$\mathrm{Ind}_\Gamma(w) =$$

$$= \boxed{N_{g_w} = \#\text{ of times that } f(z) = w \text{ in } D_r(a) \text{ with multiplicities.}}$$

Note 3 If w_1, w_2 are in the same connected component V of $\mathbb{C} \setminus \Gamma^*$, then

$$\mathrm{Ind}_\Gamma(w_1) = \mathrm{Ind}_\Gamma(w_2) \quad \text{so} \quad \#\{z \in V \mid f(z) = w_1\} = \#\{z \in V \mid f(z) = w_2\}$$

and f assumes the values w_1, w_2 equally often inside γ^*.

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Corollary 4 Suppose Ω is a domain and $f \in \mathcal{H}(\Omega)$. Assume $\boxed{g(z) = f(z) - w_0}$ has a zero of order m at z_0. Then, $\exists \varepsilon_0 > 0$ such that $\forall 0 < \varepsilon \leq \varepsilon_0$, $\exists \delta = \delta(\varepsilon) > 0$ such that

$$a \in D_\delta(w_0) \Rightarrow \text{ the equation } f(z) = a \text{ has exactly } m \text{ solutions in } D_\varepsilon(z_0)$$

up to multiplicity.
Furthermore for $a \neq w_0$ and ε small enough, all m solutions are distinct.

Picture

proof 1.) There are m solutions: Choose $\varepsilon_0 > 0$ such that $\overline{D_{\varepsilon_0}(z_0)} \subset \Omega$ and z_0 is the only solution to $g(z) = f(z) - w_0$ in $\overline{D_{\varepsilon_0}(z_0)}$. (Zeros are isolated)
Fix $0 < \varepsilon < \varepsilon_0$ and $\gamma(t) = z_0 + \varepsilon e^{it}$, $t \in [0, 2\pi]$. Let $\Gamma = f \circ \gamma$. Note $w_0 \notin \Gamma^*$ by our choice of ε_0.
Choose $\delta > 0$ such that

$$D_\delta(w_0) \cap \Gamma^* = \varnothing.$$

Then $D_\delta(w_0)$ lies in a single component of $\mathbb{C} \setminus \Gamma^*$. Thus, by **Note 3** f assumes all values $w \in D_\delta(w_0)$ equally often. Since $f(z) = w_0$ has m solution, so does $f(z) = w$ for all $w \in D_\delta(w_0)$.

2.) For $a \neq w_0$ there are m distinct solutions:

Claim: $\exists 0 < \varepsilon_1 \leq \varepsilon$ such that $z \in D'_{\varepsilon_1}(z_0) \Rightarrow f'(z) \neq 0$.

If $f'(z) \neq 0$ and $f(z) = a$, then $f(z) - a$ has an isolated zero of order 1 which means that $f(z) \neq a$ in a small disk around z. Hence if we prove the claim, the solutions of $f(z) = a$ must be distinct.
proof of Claim: As g has a zero of order m in z_0 we get for the power series

$$g(z) = f(z) - w_0 = \qquad\qquad\qquad \text{and} \quad f'(z) = \qquad .$$

Case 1 $f'(z_0) = g'(z_0) \neq 0$. Then
Case 2 $f'(z_0) = g'(z_0) = 0$. As f' is holomorphic, the zeros are isolated, so there is a punctured disk $D'_{\varepsilon_1}(z_0)$, □

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Open Mapping Theorem Suppose Ω is a region and $f \in \mathcal{H}(\Omega)$. Then $f(\Omega)$ is either a point or a region.

proof If f is constant, then $f(\Omega)$ is a point. Otherwise, let $w_0 \in f(\Omega)$, say $f(z_0) = w_0$. Then $g(z) = f(z) - w_0$ has a zero of finite order at z_0.
With ϵ_0 and δ as in the proof of the corollary we have

$$D_\delta(w_0) \subset f(D_{\epsilon_0}(z_0)) \subset f(\Omega)$$

hence $f(\Omega)$ is open. It is clearly connected, hence a region. □

Maximum Principle Suppose Ω is a region and $f \in \mathcal{H}(\Omega)$ is non-constant ($f \neq c$). Then $|f|$ has no local maximum in Ω.

proof Since Ω is a region f is non-constant on any disk in Ω. Therefore it suffices to prove the result for the case where $\Omega = D$ is a disk.
Let w_0 be in $f(D)$. Then $f(D)$ is a region by the **Open Mapping Theorem**. Hence there is $\delta > 0$, such that

We use polar coordinates. If $w_0 = r_0 e^{i\cdot\varphi_0}$, let $w = (r_0 + \frac{\delta}{2})e^{i\cdot\varphi_0} \in D_\delta(w_0)$. Then

$$|w| =$$

Hence $|f|$ can not have a maximum in D. □