Math 103: Measure Theory and Complex Analysis
Fall 2018

09/12/18

Lecture 1

Part I - MEASURE THEORY

Chapter 1 - σ algebras and measures

Outline: To give a subset of a set X a weight or measure we must restrict ourselves to "good" subsets of X or elements of $\mathcal{P}(X)$. Such a collection $\mathcal{M} \subset \mathcal{P}(X)$ is called a σ algebra.

Chapter 1.1 - Review - Riemann integral

Outline: 1.) A function f is **Riemann integrable** if it can be "approximated" by step functions. These functions are defined by subdividing the domain.
2.) The Riemann integral does not have good convergence properties. We should look for a better way of defining integration.

Definition 1 (Partitions) A **partition** or **subdivision** $\mathcal{P}$ of an interval $[a, b]$ is a finite sequence of points $\mathcal{P} = \{(t_k)_{k=0,..,n}\}$, such that

$$a = t_0 < t_1 < t_2 < \ldots < t_{n-1} < t_n = b.$$

- We call an interval (t_k, t_{k+1}) a **subinterval** of the partition $\mathcal{P}$. We call the width $w_{\mathcal{P}}$ of the largest subinterval

  $w_{\mathcal{P}} = \|\mathcal{P}\| = \max\{|t_{k+1} - t_k|, \text{ where } k \in \{0, 1, 2, \ldots, n-1\}\}$ the **mesh** or **norm** of $\mathcal{P}$.

- If for two partitions $\mathcal{P}_1, P_2$ of $[a, b]$ we have that $\mathcal{P}_1 \subset P_2$. Then $\mathcal{P}_2$ is called a **refinement** of $\mathcal{P}_1$.

Example Draw a partition $\mathcal{P}$ of the interval $[0, 10]$ and estimate its norm. Then find a refinement of $\mathcal{P}$. Given two partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ is there always a common refinement?

To define the Riemann integral we first create step functions based on a partition $\mathcal{P}$ that give an upper and lower bound on the area under the graph of f.

Defintion 2 (upper and lower Riemann sums) Let $f : [a, b] \to \mathbb{R}$ be a bounded function and $\mathcal{P} = \{(t_k)_{k=0,..,n}\}$ be a partition of $[a, b]$,i.e.

$$a = t_0 < t_1 < t_2 < \ldots < t_{n-1} < t_n = b.$$

We define two step functions $f^U, f_L : [a, b] \to \mathbb{R}$ associated to f and $\mathcal{P}$ in the following way

$$\begin{aligned} M_k &= \sup\{f(x), x \in (t_{k-1}, t_k)\} \quad \text{and} \quad f^U(x) = M_k \quad \text{for all} \quad x \in (t_{k-1}, t_k) \\ m_k &= \inf\{f(x), x \in (t_{k-1}, t_k)\} \quad \text{and} \quad f_L(x) = m_k \quad \text{for all} \quad x \in (t_{k-1}, t_k). \end{aligned}$$

If the partition $\mathcal{P}$ is important we will write $f^U_{\mathcal{P}}$ for f^U and $f_{L,\mathcal{P}}$ for f_L.
Finally the **Riemann sums** of f with respect to $\mathcal{P}$ are the integrals

$$\begin{aligned} \mathcal{U}(f, \mathcal{P}) &= \sum_{k=1}^{n} M_k \cdot (t_k - t_{k-1}) = \int_a^b f^U(x)\, dx \;\textbf{(upper sum)} \quad \text{and} \\ \mathcal{L}(f, \mathcal{P}) &= \sum_{k=1}^{n} m_k \cdot (t_k - t_{k-1}) = \int_a^b f_L(x)\, dx \;\textbf{(lower sum)} \end{aligned}$$

Note If $\mathcal{P} = \{(t_k)_{k=0,..,n}\}$ is a partition of $[a, b]$, then we are not interested in the values of the step function f on the points $(t_k)_{k=0,..,n}$ of the partition. This is because for integration it does not matter which values the the function takes on this finite number of points.

Example Sketch a continuous function f in the interval $[0, 10]$. Using your partition $\mathcal{P}$ from the previous example, sketch f^U and f_L and estimate the integrals $\mathcal{U}(f, \mathcal{P})$ and $\mathcal{L}(f, \mathcal{P})$.

Using this approximation with step functions we can try to find the "best approximating" f^U and f_L by varying and refining the partition.

Definition 3 (Upper and lower Riemann integral) Let $f : [a,b] \to \mathbb{R}$ be a bounded function. The **Riemann integrals** of f are

$$\mathcal{R}\overline{\int_a^b} f(x)\ dx = \inf\{\mathcal{U}(f,\mathcal{P}) = \int_a^b f^U_{\mathcal{P}}(x)\ dx,\ \ \mathcal{P} \text{ partition of } [a,b]\} \quad \textbf{(upper integral)}$$
$$\mathcal{R}\underline{\int_a^b} f(x)\ dx = \sup\{\mathcal{L}(f,\mathcal{P}) = \int_a^b f_{L,\mathcal{P}}(x),\ \ \mathcal{P} \text{ partition of } [a,b]\} \quad \textbf{(lower integral)} \quad .$$

Finally we say that a function f is integrable if the upper and lower Riemann integral coincide. This means that the function can be approximated by greater and lower step functions such that the corresponding integrals exists and are equal.

Defintion 4 (Riemann integrable functions) Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Then f is **(Riemann) integrable** on the interval $[a,b]$ if

$$\mathcal{R}\overline{\int_a^b} f(x)\ dx = L = \mathcal{R}\underline{\int_a^b} f(x)\ dx.$$

In this case we write $L = \mathcal{R}\int_a^b f(x)\ dx$.
The set of **Riemann integrable** functions on $[a,b]$ is denoted by $\mathcal{R}([a,b])$.
Note: For any partition $\mathcal{P}$ of $[a,b]$ we have that

$$\int_a^b f_L(x)\ dx \le \int_a^b f^U(x)\ dx \quad \text{hence} \quad \mathcal{R}\underline{\int_a^b} f(x)\ dx \le \mathcal{R}\overline{\int_a^b} f(x)\ dx.$$

Examples 5

a) **(infinite comb)** Let $f : [0,1] \to \mathbb{R}$ be the function, such that

$$f(x) = \begin{cases} 0 & \text{if} \quad x \in \mathbb{R}\setminus\mathbb{Q} \\ 1 & \quad\quad\ x \in \mathbb{Q}\,. \end{cases}$$

Then f integrable on $[0,1]$.

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b) **(popcorn function)** Let $g : [0, 1) \to \mathbb{R}$ be the function, such that

$$g(x) = \begin{cases} 0 & \text{if} \quad x \in \mathbb{R} \setminus \mathbb{Q} \\ \frac{1}{q} & \quad\quad x = \frac{p}{q} \in \mathbb{Q}, \frac{p}{q} \text{ in lowest term} \end{cases}$$

Sketch the popcorn function for $q = 2, 3, 4, 5$.

Then g integrable on $[0, 1]$.

Definition 6 A subset $S \subset \mathbb{R}$ has **measure zero** if for all $\epsilon > 0$ there are open intervals $(I_n(\epsilon))_{n \in \mathbb{N}}$ such that

a) $S \subset \bigcup_{n \in \mathbb{N}} I_n(\epsilon)$.

b) $\sum_{n \in \mathbb{N}} \ell(I_n(\epsilon)) \leq \epsilon$, where $\ell((a, b)) = b - a$.

We would like to define integration in a way such that the functions in these two previous examples have integral zero. More generally we would like to have that countable subsets of $\mathbb{R}$ have measure zero and define integration such that the integral over these sets is zero.

Example 7 Let C be a countable subset of $\mathbb{R}$. Then C has measure zero.

proof Idea: We put "small enough" intervals around every point of C.
Fix $\epsilon > 0$. Since C is countable, $C = (c_n)_{n \in \mathbb{N}} = \{c_1, c_2, c_3, \ldots\}$. We set

Note 8 The countable union of countable sets has measure zero (see **HW 1**).

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Why should we not settle for the Riemann integral?

- The integral over countable subsets of $\mathbb{R}$ is not always zero.
- The definition is restricted to bounded functions and domains.
- $\mathcal{R}([a, b[)$ does not have good convergence properties.

Example 9 Consider a sequence of functions $(f_n)_{n\in\mathbb{N}}$, such that $f_n : [0, 1] \to \mathbb{R}$, such that for all $n \in \mathbb{N}$

$$f_n \text{ continuous} \quad \text{and} \quad 0 \leq f_n(x) \leq 1 \quad \text{for all} \quad x \in [0, 1]$$

If $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in [0, 1]$. Is it true that

$$\lim_{n\to\infty} \mathcal{R} \int_0^1 f_n(x)\, dx = 0 \quad ?$$

This is in fact true, but very hard to prove. However this will be a simple result in our new integration method using measure theory.

From Riemann to Lebesque

Idea: We define integration the other way round. We look at the set I, such that $f(I)$ in an interval $[y_1, y_2]$ on the y-axis. Then

$$y_1 \cdot m(I) \leq \int_I f \leq y_2 \cdot m(I)$$

Refining the intervals now on the y-axis we get an estimate of the integral. Indeed we can define integration this way. However to this end we have to define the measure m properly. If I is an interval, we would have $m(I) = \ell(I)$.

Example

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In general we would like to have a measure $m : \mathcal{P}(\mathbb{R}) \to \mathbb{R}$, such that

a) $m(\emptyset) = 0$

b) $m(I) = \ell(I)$ for an interval $I \subset \mathbb{R}$.

c) $m(\biguplus_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} m(A_k)$.

d) $m(A + x) = m(A)$ for all $x \in \mathbb{R}$ (m is invariant under translation).

Remark 1.) Unfortunately this is not possible, as $\mathcal{P}(\mathbb{R})$ is too complex. This is a consequence of **Vitali's Theorem**.
2.) In c) we want countable additivity. If we have additivity for finite sets only, then we can not pass to limits. If we take uncountable additivity, then if we have for $p \in \mathbb{R}$ that $m(p) = 0$, then $m(\mathbb{R}) = 0$ (using d)).

Chapter 1.2 - σ algebras

Outline To define a measure we have to use the "right" subsets of X.

Definition 1 (σ algebra) Let X be a set, a collection $\mathcal{M} \subset \mathcal{P}(X)$ of subsets of X is called a σ **algebra** if

a) $X \in \mathcal{M}$.

b) $A \in \mathcal{M} \Rightarrow A^c = X \backslash A \in \mathcal{M}$ ($\mathcal{M}$ is closed under complements).

c) $(A_k)_{k=1,\ldots,\infty} \subset \mathcal{M} \Rightarrow \bigcup_{k=1}^{\infty} A_k \in \mathcal{M}$ ($\mathcal{M}$ is closed under countable unions).

In this case $(X, \mathcal{M})$ is called a **measurable space**.