Math 103: Measure Theory and Complex Analysis
Fall 2018

10/19/18

Lecture 16

Chapter 2.7. - Modes of convergence

Outline Given a measure space $(X, \mathcal{M}, \mu)$ and a sequence of measurable functions $(f_n)_n$ the sequence can converge to a function f in different ways. In general these different modes of convergence are not compatible. We will investigate their relationship.

Motivation: Convergence in $\mathbb{R}^n$

We recall that we can equip $\mathbb{R}^n$ with the different p-norms: If $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, then

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$$

If $(x_k)_k \subset \mathbb{R}^n$ is a sequence of vectors in $\mathbb{R}^n$, such that $x_k = (x_{k1}, x_{k2}, \dots, x_{kn})$ we usually we define

$$\lim_{k\to\infty} x_k = x \Leftrightarrow \lim_{k\to\infty} \|x_k - x\|_2 = 0. \qquad (*)$$

This is equivalent to $\lim_{k\to\infty} x_{ki} = x_i$ for all $i \in \{1, 2, \dots, n\}$.

In fact we could take any p-norm in (*) as in this case the convergence is independent of the norm we choose. This is due to the fact that $\mathbb{R}^n$ is a finite dimensional vector space and is no longer true for infinite dimensional vector spaces, like function spaces.

Example 1 A sequence in $\mathbb{R}^2$. For all these norms we have:

$$\lim_{k\to\infty} \|x_k - x\| = 0 \overset{v_k = x_k - x}{\Leftrightarrow} \lim_{k\to\infty} \|v_k - \overrightarrow{0}\| = 0.$$

Write down the ϵ definition of convergence to $(0, 0)$ for a sequence and sketch the convergence of such a sequence with respect to different norms.

Definition 2 (Modes of convergence) Let $(X, \mathcal{M}, \mu)$ be a measure space, $(f_n)_n : X \to \mathbb{C}$ be a sequence of measurable functions on X and f be a measurable function on X. We say that

a) $f_n \xrightarrow[n\to\infty]{\text{pointwise}} f \overset{\text{Def.}}{\iff} f_n(x) \underset{n\to\infty}{\longrightarrow} f(x)$ for all $x \in X$.
Picture Sketch a sequence $(f_n)_n$ that converges to the zero function $f \equiv 0$.

b) $f_n \xrightarrow[n\to\infty]{\text{almost everywhere}} f \overset{\text{Def.}}{\iff} f_n(x) \underset{n\to\infty}{\longrightarrow} f(x)$ for all $x \in N^c$ where $\mu(N) = 0$.
Picture Sketch a sequence $(f_n)_n$ that converges to the zero function $f \equiv 0$.

c) $f_n \xrightarrow[n\to\infty]{\text{uniformly}} f \overset{\text{Def.}}{\iff} \forall \epsilon > 0\ \exists N(\epsilon) \in \mathbb{N}$, such that

$$|f_n(x) - f(x)| < \epsilon \text{ for all } x \in X \text{ and for all } n \geq N(\epsilon).$$

Picture Sketch a sequence $(f_n)_n$ that converges to the zero function $f \equiv 0$.

d) $f_n \xrightarrow[n\to\infty]{L^1} f \overset{\text{Def.}}{\iff} \|f_n - f\|_1 := \int_X |f_n - f| \, d\mu \underset{n\to\infty}{\longrightarrow} 0.$

Question Do we always have $f_n \xrightarrow[n\to\infty]{\text{x)}} f \Leftrightarrow f_n - f \xrightarrow[n\to\infty]{\text{x)}} 0$?

Question What is the relationship between these different types of convergence?
Answer It is complicated.

We recall the LDCT which we rephrase in terms of the previous modes of convergence:

Theorem 3 (Lebesque's Dominated Convergence Theorem (LDCT))
Let $(f_n)_{n\in\mathbb{N}} : (X, \mathcal{M}) \to \mathbb{C}$ be a sequence of measurable functions on X such that

a) $f_n \xrightarrow[n\to\infty]{\text{a. e.}} f$.

b) $|f_n| \leq g$ almost everywhere for all $n \in \mathbb{N}$, where $g : (X, \mathcal{M}) \to \mathbb{R}_0^+$ and $g \in L^1(\mu)$.

Then $f \in L^1(\mu)$ and $f_n \xrightarrow[n\to\infty]{L^1} f$.

We now show what is not true:

Counter-examples 4

a) $f_n \xrightarrow[n\to\infty]{\text{a. e.}} f \not\Rightarrow f_n \xrightarrow[n\to\infty]{L^1} f$.
Sliding door:

b) $f_n \xrightarrow[n\to\infty]{\text{uniformly}} f \not\Rightarrow f_n \xrightarrow[n\to\infty]{L^1} f$.
Ebb tide:

c) $f_n \xrightarrow[n\to\infty]{L^1} f \not\Rightarrow f_n \xrightarrow[n\to\infty]{\text{a. e.}} f$

Typewriter sequence: Let $X = [0,1]$ and set

$$\begin{aligned} f_1 &= \mathbb{1}_{[0,1]} \\ f_2 &= \mathbb{1}_{[0,\frac{1}{2}]}\ ,\ f_3 = \mathbb{1}_{[\frac{1}{2},1]} \\ f_4 &= \mathbb{1}_{[0,\frac{1}{4}]}\ ,\ f_5 = \mathbb{1}_{[\frac{1}{4},\frac{1}{2}]}\ ,\ f_6 = \mathbb{1}_{[\frac{1}{2},\frac{3}{4}]}\ ,\ f_7 = \mathbb{1}_{[\frac{3}{4},1]} \\ &\text{etc.} \end{aligned}$$

Then

$$\int_{[0,1]} |f_n - 0|\, d\mu =$$

This means that $f_n \xrightarrow[n\to\infty]{L^1} f$, but $(f_n)_n$ does not converge almost everywhere.

Question What does L^1 convergence imply?

If $\mu(A) > 0$ and for all $x \in A$ we have that $|f_n(x) - f(x)| \geq c > 0$, for all $n \in \mathbb{N}$ then

$$\|f_n - f\|_1 =$$

Hence if $f_n \xrightarrow[n\to\infty]{L^1} f$ then

$$\mu(A \mid |f_n(x) - f(x)| \geq c > 0 \text{ for all } x \in A \text{ and } n \in \mathbb{N}\} = 0.$$

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Convergence in measure

There is still another important type of convergence.

Definiton 5 Let $(X, \mathcal{M}, \mu)$ be a measure space, $(f_n)_n : X \to \mathbb{C}$ be a sequence of measurable functions on X and f be a measurable function on X. We say that $(f_n)_n$ **converges in measure** to a measurable function $f : X \to \mathbb{C}$ if for all $\epsilon > 0$

$$\lim_{n\to\infty} \mu(x \in X \mid |f_n(x) - f(x)| \geq \epsilon\} = 0.$$

In this case we write shortly $f_n \xrightarrow[n\to\infty]{\mu} f$.

Example:

Proposition 6 Let $(X, \mathcal{M}, \mu)$ be a measure space, then $f_n \xrightarrow[n\to\infty]{L^1} f$ implies $f_n \xrightarrow[n\to\infty]{\mu} f$.

proof Note that for $A(\epsilon, n) = \mu(x \in X \mid |f_n(x) - f(x)| \geq \epsilon\}$ we have that

$$\|f_n - f\|_1 =$$

Hence using the two definitions of convergence we obtain:

We can not expect convergence in measure to imply convergence a. e.. However:

Proposition 7 Let $(X, \mathcal{M}, \mu)$ be a measure space. If $f_n \xrightarrow[n\to\infty]{\mu} f$ then there is a subsequence $(f_{n_k})_k$ such that $f_{n_k} \xrightarrow[k\to\infty]{\text{a. e.}} f$.

proof: HW 3