**Math 103: Measure Theory and Complex Analysis**
**Fall 2018**

10/12/18

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**Lecture 13**

**Theorem 4 (Monotone Class Lemma (MCL))** If $\mathcal{A} \subset \mathcal{P}(X)$ is an algebra then

$$\mathcal{M}(\mathcal{A}) = \langle \mathcal{A} \rangle = \mathcal{C}(\mathcal{A}).$$

**proof** As every $\sigma$ algebra containing $\mathcal{A}$ is a monotone class containing $\mathcal{A}$ we know that $\mathcal{C}(\mathcal{A}) \subset \mathcal{M}(A)$. It remains to show the inverse direction.
We start by constructing a monotone class for each $\boxed{E \in \mathcal{C} = \mathcal{C}(\mathcal{A})}$: for $E \in \mathcal{C}$ set

$$\mathcal{C}(E) := \{F \in \mathcal{C} \mid E \backslash F \in \mathcal{C}\,,\ F \backslash E \in \mathcal{C} \ \text{ and } F \cap E \in \mathcal{C}\}.$$

**Picture**

We observe

1.) As $\emptyset \in \mathcal{A} \subset \mathcal{C} = \mathcal{C}(\mathcal{A})$ we have that $\emptyset \in \mathcal{C}$ and $E \in \mathcal{C}$.

2.) By definition: $F \in \mathcal{C}(E) \Leftrightarrow$

3.) $\mathcal{C}(E)$ is a monotone class as $\mathcal{C}$ is a monotone class (check it).

Since $\mathcal{A}$ is an algebra, we know that for all $\boxed{E \in \mathcal{A}}$: $\mathcal{A} \subset \mathcal{C}(E)$: for $A, E \in \mathcal{A}$ we know that

$A \cap E =$ $\qquad$ $E \backslash A =$ $\qquad$ $A \backslash E =$

With 3.) this implies that for all $\boxed{E \in \mathcal{A}}$

4.) $\mathcal{C} \in \mathcal{C}(E)$ or $F \in \mathcal{C}(E)$ for all $F \in \mathcal{C}$.

By 2.) this implies that $E \in \mathcal{C}(F)$ for all $E \in \mathcal{A}$ and $F \in \mathcal{C}$. Hence

This means that if $E, F \in \mathcal{C}$ then

$$E \backslash F \in \mathcal{C}\,,\ F \backslash E \in \mathcal{C} \ \text{ and } F \cap E \in \mathcal{C}\,.$$

Since $X, \emptyset \in \mathcal{A} \subset \mathcal{C}$ we see that $\mathcal{C}$ is an algebra (check the conditions).

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It remains to show that $\mathcal{C}$ is also a $\sigma$ algebra. Then it is a $\sigma$ algebra containing $\mathcal{A}$ and hence $\mathcal{M}(\mathcal{A}) \subset \mathcal{C}$. To show the missing closure under countable unions we observe: if $(E_i)_{i\in\mathbb{N}} \subset \mathcal{C}$, then as $\mathcal{C}$ is an algebra

$$\bigcup_{k=1}^{i} E_k \in \mathcal{C} \;\Rightarrow\; \bigcup_{i\in\mathbb{N}} E_i =$$

as $\mathcal{C}$ is a monotone class. In total we have shown our lemma. □

**Aim:** Using the MCL we can now study the measurability of slice functions in more detail.

**Proposition 5** Suppose $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are $\sigma$ finite measure spaces. If $E \in \mathcal{M} \otimes \mathcal{N}$ then the maps

$$x \to \nu(E_x) \text{ and } y \to \mu(E_y) \text{ are measurable and} \tag{1}$$

$$\mu \times \nu(E) = \int_X \nu(E_x)\, d\mu = \int_Y \mu(E_y)\, d\nu. \tag{2}$$

**proof** Suppose first that $\mu < \infty$ and $\nu < \infty$. Let $\mathcal{C}$ be the collection of sets for which the conclusions of the proposition hold:

$$\mathcal{C} = \{E \in \mathcal{M} \otimes \mathcal{N} \mid E \text{ satisfies } (1), (2)\}.$$

We know that

- $\mathcal{R} \subset \mathcal{C}$: if $A \times B = R$ is a rectangle then clearly

  $$\nu((A \times B)_x) = \qquad\qquad \text{and} \quad \int_X \nu((A \times B)_x)\, d\mu =$$

  As the same reasoning applies to the other integral, we have that any rectangle $R$ is in $\mathcal{C}$.

- $\mathcal{A} \subset \mathcal{C}$: every element in $\mathcal{A}$ can be written as a disjoint union of elements in $\mathcal{R}$. This implies that $\mathcal{A} \subset \mathcal{C}$.

As the algebra $\mathcal{A} \subset \mathcal{C}$ it therefore suffices to show that $\mathcal{C}$ is a monotone class.
Then $\boxed{\langle \mathcal{A} \rangle = \mathcal{M} \otimes \mathcal{N} = \mathcal{C}(\mathcal{A}) \subset \mathcal{C}}$. This will follow from the MCT.
So let $(E_i)_{i\in\mathbb{N}} \subset \mathcal{C}$ with $E_i \subset E_{i+1}$ and $E = \bigcup_{i\in\mathbb{N}} E_i$. Let

$$\boxed{f_n(y) = \mu((E_n)_y)} \text{ and } \boxed{f(y) = \mu(E_y)}.$$

Then $f_n \geq 0$ is measurable and the sequence $(f_n)_n$ is an increasing sequence of positive functions, such that $\lim_{n\to\infty} f_n = f$. Hence we can apply the MCT and obtain

$$\int_Y \mu(E_y)\, d\nu =$$

Similarly $\int_X \nu(E_x)\, d\mu = \mu \times \nu(E)$, so $E \in \mathcal{C}$.
To prove the statement for countable sections of sets on $\mathcal{C}$ we proceed in a similar fashion and construct a decreasing sequence of functions. We can then argue with the DCT.
In total we have proven that if $\mu < \infty$ and $\nu < \infty$ then $\mathcal{C}$ is a monotone class, which was the missing piece of our proof.
In general, if $\mu$ or $\nu$ are not finite, we use the fact that $X \times Y = \bigcup_{i\in\mathbb{N}} A_i \times B_i$, where

$$A_i \times B_i \in \mathcal{R}\ ,\ A_i \times B_i \subset A_{i+1} \times B_{i+1} \text{ for all } i \in \mathbb{N} \quad \text{and} \quad \mu \times \nu(A_i \times B_i) < \infty.$$

For $E \in \mathcal{M} \otimes \mathcal{N}$ we can then write

$$\mu \times \nu(A_i \times B_i) =$$

The statement then follows again with the MCT. □

**Notation** Given a measure space $(X, \mathcal{M}, \mu)$ we set

$$L^+(X, \mathcal{M}, \mu) = L^+(\mu) = \{f : X \to [0, \infty] \mid f \text{ measurable }\}$$
$$\mathcal{L}^1(X, \mathcal{M}, \mu) = \mathcal{L}^1(\mu) = \{f : X \to \mathbb{C} \mid \int_X |f|\, d\mu < \infty\}$$

Finally we have

**Theorem 6 (Fubini-Tonelli)** Suppose $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are $\sigma$ finite measure spaces. Then

a) **(Tonelli)** Suppose $f \in L^+(\mu \times \nu)$ then

   i) $f_x \in L^+(\mu)$ and $f_y \in L^+(\nu)$ for all $x \in X$ and $y \in Y$.

   ii) If $g(x) = \int_Y f_x\, d\nu$ and $h(y) = \int_X f_y\, d\mu$. Then $g \in L^+(\mu)$ and $h \in L^+(\nu)$.

   iii) $\displaystyle\int_{X\times Y} f\, d\mu \times \nu = \int_X \left(\int_Y f(x,y)\, d\mu\right) d\nu = \int_Y \left(\int_X f(x,y)\, d\nu\right) d\mu.$

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b) **(Fubini)** Suppose $f \in \mathcal{L}^1(\mu \times \nu)$ then

  i) $f_x$ and $f_y$ are measurable for all $x \in X$ and $y \in Y$.

  ii) For $\mu$ almost all $x \in X$, $f_x \in \mathcal{L}^1(\nu)$ and for $\mu$ almost all $y \in Y$, $f_y \in \mathcal{L}^1(\mu)$. If

$$g(x) = \begin{cases} \int_Y f_x \, d\nu & \text{if} \quad f_x \in \mathcal{L}^1(\nu) \\ 0 & \quad\; f_x \notin \mathcal{L}^1(\nu) \end{cases} \quad \text{and} \quad h(y) = \begin{cases} \int_X f_y \, d\mu & \text{if} \quad f_y \in \mathcal{L}^1(\mu) \\ 0 & \quad\; f_y \notin \mathcal{L}^1(\mu) \end{cases}.$$

  Then $g \in \mathcal{L}^1(\mu)$ and $h \in \mathcal{L}^1(\nu)$.

  iii) $\displaystyle\int_{X \times Y} f \, d\mu \times \nu = \int_X \left( \int_Y f(x,y) \, d\mu \right) d\nu = \int_Y \left( \int_X f(x,y) \, d\nu \right) d\mu.$

**proof Tonelli** i) If $f$ is characteristic function then the result follows from the previous proposition.

ii) If $g_n(x) = \int_Y (f_n)_x \, d\nu$ and $h_n(x) = \int_X (f_n)_y \, d\mu$ then by the monotonicity of the integral we can approximate $g$ and $f$ by the increasing sequences $(g_n)_n$ and $(h_n)_n$ of measurable functions, such that $\lim_{n\to\infty} g_n = g$ and $\lim_{n\to\infty} h_n = h$. Then $g \in L^+(\nu)$ and $h \in L^+(\mu)$ and ii) holds.
iii) By the MCT

$$\int_X g \, d\mu =$$

By symmetry $\displaystyle\int_Y h \, d\nu = \int_{X\times Y} f \, d(\mu \times \nu)$ which concludes the proof of a) □

**Fubini** If $f \in \mathcal{L}^1(\mu \times \nu)$, we can apply Tonelli to $\mathrm{Re}(f)^\pm$ and $\mathrm{Im}(f)^\pm$. So we may assume that $f \in \mathcal{L}^1(\mu \times \nu) \cap L^+(\mu \times \nu)$. Then $f_x$ and $f_y$ are measurable, as is $\tilde{g}(x) = \int_Y f_x \, d\nu$ and $\int_X \tilde{g} \, d\mu < \infty$.
Thus for $N = \{x : \tilde{g}(x) = \infty\}$ is a $\mu$-null set so $g$ is $\mu$ measurable and $\tilde{g} = g$ almost everywhere. In particular $g \in \mathcal{L}^1(\mu)$ and

$$\int_X g \, d\mu = \int_{X\times Y} f \, d(\mu \times \nu).$$

The rest follows by symmetry. □

**Remark** In practice Fubini is usually used to reverse the order of integration to obtain simpler integrals in a formula like

$$\int_X \int_Y f(x,y) \, d\nu \, d\mu$$

First we show that $f$ is $\mathcal{M} \otimes \mathcal{N}$ measurable. Then we apply Tonelli's theorem to $\int_X \int_Y |f(x,y)| \, d\nu \, d\mu$ to see that $f \in \mathcal{L}^1(\mu \times \nu)$. Then Fubini's theorem applies.

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