How Many Borel Sets are There?

Object. This series of exercises is designed to lead to the conclusion that if $\mathcal{B}_{\mathbf{R}}$ is the σ -algebra of Borel sets in \mathbf{R} , then

 $\operatorname{Card}(\mathcal{B}_{\mathbf{R}}) = \mathfrak{c} := \operatorname{Card}(\mathbf{R}).$

This is the conclusion of problem 4. As a bonus, we also get some insight into the "structure" of $\mathcal{B}_{\mathbf{R}}$ via problem 2. This just scratches the surface. If you still have an itch after all this, you want to talk to a set theorist. This treatment is based on the discussion surrounding [1, Proposition 1.23] and [2, Chap. V §10 #31].

For these problems, you will need to know a bit about well-ordered sets and transfinite induction. I suggest [1, §0.4] where transfinite induction is [1, Proposition 0.15]. Note that by [1, Proposition 0.18], there is an uncountable well ordered set Ω such that for all $x \in \Omega$, $I_x := \{ y \in \Omega : y < x \}$ is countable. The elements of Ω are called the *countable ordinals*. We let $1 := \inf \Omega$. If $x \in \Omega$, then $x + 1 := \inf \{ y \in \Omega : y > x \}$ is called the *immediate successor* of x. If there is a $z \in \Omega$ such that z + 1 = x, then z is called the *immediate predecessor* of x. If x has no immediate predecessor, then x is called a *limit ordinal*.¹

1. Show that $\operatorname{Card}(\Omega) \leq \mathfrak{c}$. (This follows from [1, Propositions 0.17 and 0.18]. Alternatively, you can use transfinite induction to construct an injective function $f : \Omega \to \mathbf{R}$.)²

2. If X is a set, let $\mathscr{P}(X)$ be the set of subsets of X — i.e., $\mathscr{P}(X)$ is the *power set* of X. Let $\mathscr{E} \subset \mathscr{P}(X)$. The object of this problem is to give a "concrete" description of the σ -algebra $\mathscr{M}(\mathscr{E})$ generated by \mathscr{E} . (Of course, we are aiming at describing the Borel sets in **R** which are generated by the collection \mathscr{E} of open intervals.) For convenience, we assume that $\emptyset \in \mathscr{E}$.

Let

$$\mathscr{E}^c := \{ E^c : E \in \mathscr{E} \} \text{ and } \mathscr{E}_\sigma = \{ \bigcup_{i=1}^\infty E_i : E_i \in \mathscr{E} \}.$$

(Note, I just mean that \mathscr{E}_{σ} is the set of sets formed from countable unions of elements of \mathscr{E} . Since $\emptyset \in \mathscr{E}, \mathscr{E} \subset \mathscr{E}_{\sigma}$.)

¹The set of countable ordinals has a rich structure. We let 2 := 1 + 1, and so on. The set $\{n \in \mathbb{N}\} \subset \Omega$ is countable, and so has a supremum ω (see [1, Proposition 0.19]). Then there are ordinals $\omega + 1$, $\omega + 2$, ..., 2ω , $2\omega + 1$, ..., ω^2 , $\omega^2 + 1$, ..., ω^{ω} , and so on.

²The issue of whether or not $Card(\Omega) = \mathfrak{c}$ is the *continuium hypothesis*, and so is independent of the usual (ZFC) axioms of set theory.

We let $\mathscr{F}_1 := \mathscr{E} \cup \mathscr{E}^c$. If $x \in \Omega$, and if x has an immediate predecessor y, then we set

$$\mathscr{F}_x := (\mathscr{F}_y)_\sigma \cup \left((\mathscr{F}_y)_\sigma \right)^c$$

If x is a limit ordinal, then we set

$$\mathscr{F}_x := igcup_{y < x} \mathscr{F}_y$$

We aim to show that

$$\mathscr{M}(\mathscr{E}) = \bigcup_{x \in \Omega} \mathscr{F}_x \tag{\dagger}$$

- (a) Observe that $\mathscr{F}_1 \subset \mathscr{M}(\mathscr{E})$.
- (b) Show that if $F_y \subset \mathscr{M}(\mathscr{E})$ for all y < x, then $F_x \subset \mathscr{M}(\mathscr{E})$. Then use transfinite induction to conclude that $\mathscr{F}_x \subset \mathscr{M}(\mathscr{E})$ for all $x \in \Omega$.
- (c) Show that the right-hand side of (†) is closed under countable unions.
- (d) Conclude that $\bigcup_{x\in\Omega} \mathscr{F}_x$ is a σ -algebra, and that (\dagger) holds.

3. Recall that if A and B are sets, then $\prod_{a \in A} B$ is simply the set of functions from A to B. For reasons that are unclear to me, this set is usually written B^A . Notice that $\prod_{i=1}^{\infty} B = \prod_{i \in \mathbf{N}} B$ is just the collection of sequences in B. Notice also that $\operatorname{Card}(B^A)$ depends only on $\operatorname{Card}(A)$ and $\operatorname{Card}(B)$.

(a) Check that

$$\prod_{i=1}^{\infty} \left(\prod_{j=1}^{\infty} B\right) = \prod_{(i,j)\in\mathbf{N}\times\mathbf{N}} B.$$
(*)

Thus the cardinality of either side of (*) is the same as $\prod_{i=1}^{\infty} B$.

(b) Use these observations together with the fact that $\operatorname{Card}\left(\prod_{i=1}^{\infty} \{0,1\}\right) = \mathfrak{c} := \operatorname{Card}(\mathbf{R})$ (which follows from [1, Proposition 0.12]) to show that

$$\operatorname{Card}\left(\prod_{i=1}^{\infty}\mathbf{R}\right) = \mathfrak{c}$$

(c) Show that if $\operatorname{Card}(\mathscr{E}) = \mathfrak{c}$, then $\operatorname{Card}(\mathscr{E}_{\sigma}) = \mathfrak{c}$.

4. Let $\mathcal{B}_{\mathbf{R}}$ be the σ -algebra of Borel sets in \mathbf{R} . In [1, Proposition 0.14(b)], it is shown that if $\operatorname{Card}(A) \leq \mathfrak{c}$ and if $\operatorname{Card}(Y_x) \leq \mathfrak{c}$ for all $x \in A$, then $\bigcup_{x \in A} Y_x$ has cardinality bounded by \mathfrak{c} . By following the given steps, use this observation, as well as problems 2 and 3, to show that

$$\operatorname{Card}(\mathcal{B}_{\mathbf{R}}) = \mathfrak{c}.$$
 (‡)

- (a) Let \mathscr{E} be the collection of open intervals (including the empty set) in **R**. Then $\operatorname{Card}(\mathscr{E}) = \mathfrak{c}$.
- (b) $\mathcal{B}_{\mathbf{R}} = \mathscr{M}(\mathscr{E}).$
- (c) Define \mathscr{F}_x as in problem 2. Use transfinite induction and problem 3 to prove that $\operatorname{Card}(F_x) = \mathfrak{c}$ for all $x \in \Omega$.
- (d) Use problem 2 to conclude that $\mathscr{M}(\mathscr{E}) = \mathscr{B}_{\mathbf{R}}$ has the cardinality claimed in $(\ddagger).^4$

References

- [1] Gerald B. Folland, *Real analysis*, Second, John Wiley & Sons Inc., New York, 1999. Modern techniques and their applications, A Wiley-Interscience Publication.
- [2] Anthony W. Knapp, *Basic real analysis*, Cornerstones, Birkhäuser Boston Inc., Boston, MA, 2005. Along with a companion volume *Advanced real analysis*.

⁴It is my *understanding* that the classes \mathscr{F}_x are all distinct; that is, $\mathscr{F}_x \subsetneq \mathscr{F}_y$ if x < y in Ω . But I don't have a reference or a proof at hand.