Math 73/103 Assignment Four
Due Date TBA

1. Let ν be a complex measure on $(X, \mathfrak{M})$.

   (a) Show that there is a measure μ and a measurable function $\varphi : X \to \mathbf{C}$ so that $|\varphi| = 1$, and such that for all $E \in \mathfrak{M}$,
   $$\nu(E) = \int_E \varphi \, d\mu. \tag{†}$$
   (Hint: write $\nu = \nu_1 - \nu_2 + i(\nu_3 - \nu_4)$ for measures ν_i. Put $\mu_0 = \nu_1 + \nu_2 + \nu_3 + \nu_4$. Then μ_0 will satisfy (†) provided we don't require $|\varphi| = 1$. You can then use without proof the fact that any complex-valued measurable function h can be written as $h = \varphi \cdot |h|$ with φ unimodular and measurable.)

   (b) [Optional: Do not turn in] Show that the measure μ above is unique, and that φ is determined almost everywhere $[\mu]$. (Hint: if μ' and φ' also satisfy (†), then show that $\mu' \ll \mu$, and that $\frac{d\mu'}{d\mu} = 1$ a.e. Also note that if φ' is unimodular and $E \in \mathfrak{M}$, then $E = \bigcup_{i=1}^4 E_i$ where $E_1 = \{\, x \in E : \operatorname{Re} \varphi' > 0 \,\}$, $E_2 = \{\, x \in E : \operatorname{Re} \varphi' < 0 \,\}$, $E_3 = \{\, x \in E : \operatorname{Im} \varphi' > 0 \,\}$, and $E_4 = \{\, x \in E : \operatorname{Im} \varphi' < 0 \,\}$.)

Comment: the measure μ in question 1 is called the *total variation* of ν, and the usual notation is $|\nu|$. It is defined by different methods in your text: see chapter 6. One can prove facts like $|\nu|(E) \geq |\nu(E)|$, although one doesn't always have $|\nu|(E) = |\nu(E)|$; this also proves that even classical notation can be unfortunate.

The remaining problems reprise some of the fundamental results about functions of a complex variable covered in elementary courses but not covered in chapter ten of our text [2]. Most of this material — with perhaps the exception of problem 6 — are part of the early chapters in basic texts such as Conway [1], Brown & Churchill or Saff & Snider. Feel free to sneak a peak.

Let Ω be a domain in $\mathbf{C}$ and assume that $f : \Omega \to \mathbf{C}$ is a function. Of course, we can view Ω as an open subset of $\mathbf{R}^2$ and define $u, v : \Omega \to \mathbf{R}$ by
$$u(x, y) := \operatorname{Re}\bigl(f(x + iy)\bigr) \quad \text{and} \quad v(x, y) = \operatorname{Im}\bigl(f(x + iy)\bigr)$$
We say that the *Cauchy-Riemann Equations hold at* $z_0 = x_0 + iy_0$ if the partial derivatives of u and v exist at (x_0, y_0) and
$$u_x(x_0, y_0) = v_y(x_0, y_0) \quad \text{and} \quad u_y(x_0, y_0) = -v_x(x_0, y_0). \tag{CR}$$

We often abuse notation slightly, and say that (CR) amounts to $f_y(z_0) = if_x(z_0)$. (Just to be specific, $f_x(x_0 + iy_0) := u_x(x_0, y_0) + iv_x(x_0, y_0)$.)

2. Suppose that $f'(z_0)$ exists. Show that

$$f_x(z_0) = f'(z_0) = -if_y(z_0). \tag{1}$$

Conclude that the Cauchy-Riemann equations hold at z_0 whenever $f'(z_0)$ exists. Verify (1) when $f(z) = z^2$. (Write $f'(z) = \lim_{h\to 0} \frac{1}{h}\bigl(f(z+h) - f(z)\bigr)$. If the limit exists, so does the limit when we let $h = x + i0$ be real or $h = 0 + iy$ is purely imaginary.)

3. Suppose that Ω is a *region* in $\mathbf{C}$, and that $f \in H(\Omega)$. Show that if $f'(z) = 0$ for all $z \in \Omega$, then f is constant. (You can prove for yourself or use without proof that if $u : \Omega \subset \mathbf{R}^2 \to \mathbf{R}$ is such that $u_x(x, y) = u_y(x, y)$ for all $(x, y) \in \Omega$ then u is constant — provided Ω is a region.)

4. Suppose that Ω is a region and $f \in H(\Omega)$. Show that if f is real-valued in Ω, then f is constant.

5. Suppose that Ω is a region and $f \in H(\Omega)$. Suppose that $z \mapsto |f(z)|$ is constant on Ω. Show that f must be constant. (Consider $|f(z)|^2$.)

We let f, u, v and Ω be as above. Define

$$F : \Omega \subset \mathbf{R}^2 \to \mathbf{R}^2 \quad \text{by} \quad F(x, y) = \bigl(u(x, y), v(x, y)\bigr).$$

Pretend that you remember that F is differentiable at $(x_0, y_0) \in \Omega$ if there is a linear function $L : \mathbf{R}^2 \to \mathbf{R}^2$ such that

$$\lim_{(h,k)\to(0,0)} \frac{\|F(x_0 + h, y_0 + k) - F(x_0, y_0) - L(h, k)\|}{\|(h, k)\|} = 0,$$

in which case, the partials of u and v must exist and L is given by the Jacobian Matrix

$$[L] = \begin{pmatrix} u_x(x_0, y_0) & u_y(x_0, y_0) \\ v_x(x_0, y_0) & v_y(x_0, y_0) \end{pmatrix}.$$

(Of course, here $\|(x, y)\| = \sqrt{x^2 + y^2} = |x + iy|$.)

6. Let f, F, u, v and Ω be as above. Let $z_0 = x_0 + iy_0 \in \Omega$. Show that $f'(z_0)$ exists if and only if the Cauchy-Riemann equations hold at z_0 and F is differentiable at (x_0, y_0). (Hint: if we let $z = h + ik$ and if T is given by the matrix

$$[T] = \begin{pmatrix} u_x(x_0,y_0) & -v_x(x_0,y_0) \\ v_x(x_0,y_0) & u_x(x_0,y_0) \end{pmatrix},$$

then

$$\|F(x_0+h, y_0+k) - F(x_0,y_0) - T(h,k)\| = |f(z+z_0) - f(z_0) - \omega z|,$$

where $\omega = u_x(x_0,y_0) + iv_x(x_0,y_0) = f_x(z_0)$. Then remember (1).)

Problem #6 has an important Corollary. We learn in multivariable calculus, that F is differentiable at (x_0, y_0) if the partial derivatives of u and v exist in a neighborhood of (x_0, y_0) and are continuous at (x_0, y_0). Hence we get as a Corollary of problem #6, with f, u and v defined as above, that if u and v have continuous partial derivatives in a neighborhood of (x_0, y_0) and if the Cauchy-Riemann equations hold at z_0, then $f'(z_0)$ exists. Use this observation in problem #7.

7. Define $\exp : \mathbf{C} \to \mathbf{C}$ by $\exp(x+iy) = e^x\big(\cos(y) + i\sin(y)\big)$. Show that $\exp \in H(\mathbf{C})$ and $\exp'(z) = \exp(z)$ for all $z \in \mathbf{C}$.

If Ω is open in $\mathbf{C}$ or $\mathbf{R}^2$, then we say $u : \Omega \to \mathbf{R}$ is *harmonic* if it has continuous second partial derivatives and if it is a solution to Laplace's equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{L}$$

8. Suppose that $f \in H(\Omega)$. Let $u(x,y) = \operatorname{Re}\big(f(x+iy)\big)$. Assuming u has continuous second partials, show that u is harmonic in Ω.

9. Suppose that $u : \Omega \to \mathbf{R}$ is harmonic. We say $v : \Omega \to \mathbf{R}$ is a *harmonic conjugate* for u if $f(x+iy) = u(x,y) + iv(x,y)$ defines a holomorphic function on Ω. Find all harmonic conjugates for $u(x,y) = 2xy$.

For the purposes of this assignment only, we'll call a region Ω a *SC region* if every $f \in H(\Omega)$ has an antiderivative in Ω. For example, we have shown in lecture that every convex region is a SC region. Later, I hope that we'll see that any simply connected region is a SC region. In fact, a region is a SC-region if and only if it is simply connected.

10. Suppose that Ω is a SC region and that u is harmonic in Ω. Show that u has a harmonic conjugate in Ω. (Hint: we need to find a function $f \in H(\Omega)$ such that $u = \mathrm{Re}(f)$. However, consider $g = u_x - iu_y$. Show that $g \in H(\Omega)$ and consider an anti-derivative f for g in Ω. As in problem 3, you may use without proof the fact that if $w : \Omega \to \mathbf{R}$ is continuous and $w_x \equiv 0 \equiv w_y$ in Ω, then w is constant.)

If $u = \mathrm{Re}(f)$, then $u_x = \mathrm{Re}(f')$ and $u_y = \mathrm{Re}(-if')$. Thus, it is a consequence of question #10 (and the deep result that $f \in H(\Omega)$ implies f is analytic) that every harmonic function has continuous partial derivatives of all orders.

11. Just as in question #7, we'll be fancy and write $\exp(z)$ in place of e^z. Suppose that Ω is a SC region and that $0 \notin \Omega$. Then show there is a $f \in H(\Omega)$ such that

$$\exp\bigl(f(z)\bigr) = z.$$

We call f a *branch of* $\log(z)$ *in* Ω. (Hint: start by letting f be an antiderivative of $1/z$. and recall that $\exp(z) = a$ has infinitely many solutions for all $a \neq 0$.)

12. Show that $f(z) = 1/z$ can't have an antiderivative in the punctured complex plane $\mathbf{C}^* := \mathbf{C} \setminus \{0\}$. Conclude that there is no (holomorphic) branch of $\log z$ in $\mathbf{C}^*$.

References

[1] John B. Conway, *Functions of one complex variable*, Second, Graduate Texts in Mathematics, vol. 11, Springer-Verlag, New York, 1978. MR503901 (80c:30003)

[2] Walter Rudin, *Real and complex analysis*, McGraw-Hill, New York, 1987.