Mathematics 101
Fall 2014
Homework 8 (not to hand in)

1. Let p be an odd prime, $n \geq 2$, and let $G = GL_n(\mathbb{F}_p)$, where $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ is the field with p elements.

   (a) Determine the size of G and via the first isomorphism theorem, deduce the order of the subgroup $H = SL_n(\mathbb{F}_p)$.

   (b) Show that $GL_n(\mathbb{F}_p)$ is a split extension of $SL_n(\mathbb{F}_p)$ by $\mathbb{F}_p^\times$.

   (c) Show that every Sylow p-subgroup of G is also a Sylow p-subgroup of H.

   (d) Now restrict n to be $n = 2$ or $n = 3$. For each case exhibit a Sylow p subgroup. *Hint:* If the Sylow subgroups have size p^m, then every element of a Sylow p-subgroup in annihilated by the polynomial $x^{p^m} - 1 \in \mathbb{F}_p[x]$. Note that you are in characteristic p, and you are dealing with matrices, so perhaps a little linear algebra might apply.

   (e) For each of $n = 2, 3$ determine how many Sylow p-subgroups there are in $GL_n(\mathbb{F}_p)$.

2. Semidirect products. We showed in class that $Aut(\mathbb{Z}_n) \cong \mathbb{Z}_n^\times$.

   (a) Suppose that H_1, H_2 and K are groups, $\sigma : H_1 \to H_2$ is an isomorphism, and $\psi : H_2 \to Aut(K)$ a homomorphism, so that $\varphi = \psi \circ \sigma : H_1 \to Aut(K)$ is also a homomorphism. Show that $K \rtimes_\varphi H_1 \cong K \rtimes_\psi H_2$.

   (b) Suppose that H and K are groups and $\varphi, \psi : H \to Aut(K)$ are monomorphisms with the same image in $Aut(K)$. Show that there exists a $\sigma \in Aut(H)$ such that $\psi = \varphi \circ \sigma$.

   (c) Suppose that H and K are groups, $\varphi, \psi : H \to Aut(K)$ are monomorphisms, and $Aut(K)$ is finite and cyclic. Show that φ and ψ have the same image in $Aut(K)$.

   (d) Let $p < q$ be primes with $p \mid (q-1)$. Let H and K be cyclic groups of order p and q respectively. Let $\varphi, \psi : H \to Aut(K)$ be nontrivial homomorphisms. Observing that $Aut(K)$ is cyclic, show that $K \rtimes_\varphi H \cong K \rtimes_\psi H$.

   (e) Let $p < q$ be primes. Show that up to isomorphism, there are at most two groups of order pq.

3. Let H be a group, and by H^n denote group which is the external direct product of H with itself n times. Show that the symmetric group S_n acts on H^n via $(\sigma, (h_1, \dots, h_n)) \mapsto (h_{\sigma^{-1}(1)}, \dots, h_{\sigma^{-1}(n)})$. Note that the obvious map $(\sigma, (h_1, \dots, h_n)) \mapsto (h_{\sigma(1)}, \dots, h_{\sigma(n)})$ is a right action, that is you need the inverses so that the permutation representation is a homomorphism.

   Hint: This is a bit subtle with notation; you may find the following observation useful. In set theory, Y^X denotes the set of functions $f : X \to Y$, so one can interpret H^n as

H^X where $X = \{1, 2, \dots, n\}$. Now show that the natural action of S_n on X induces an action of S_n on $H^X \cong H^n$ as suggested above.

4. Let A, B be groups and let $|B| = n$. We know that B acting on itself by left translation induces an injective homomorphism $\rho : B \to S_n$; this permutation representation is usually called the *left regular representation.* As we saw in the previous problem, S_n acts on A^n with permutation representation $\varphi : S_n \to Aut(A^n)$. The composition of φ and ρ, is a homomorphism $\varphi \circ \rho : B \to Aut(A^n)$. The *wreath product* of A by B, denoted $A \wr B$, is defined to be the semidirect product $A \wr B = A^n \rtimes_{\varphi\circ\rho} B$.

   Show (obvious) that the order of $A \wr B$ is $|A|^{|B|} \cdot |B|$. From this it is clear that $\mathbb{Z}_2 \wr \mathbb{Z}_2$ is a group of order 8. Determine its isomorphism class in part by writing out all eight elements and finding their orders.

5. The point of this exercise is to show that for p a prime, any Sylow p-subgroup of S_{p^2} is isomorphic to $\mathbb{Z}_p \wr \mathbb{Z}_p$.

   (a) First compute the cardinality of a Sylow p-subgroup of S_{p^2} (without assuming the isomorphism).

   (b) The following is outline of a proof is from Rotman's *Theory of Groups*, where a more general result is established. Let $B_0 = \{1, 2, \dots, p\}$ and let $B_i = B_0 + ip$, so that the set $\{1, 2, \dots, p^2\}$ is the disjoint union $B_0 \cup \cdots \cup B_{p-1}$. Consider the permutation $\sigma \in S_{p^2}$ which for $b_0 \in B_0$ takes

   $$\sigma(b_0 + ip) = \begin{cases} b_0 + (i+1)p & \text{if } i < p - 1, \\ b_0 & \text{if } i = p - 1. \end{cases}$$

   Verify that $B_{i+1} = \sigma(B_i)$ (with the subscripts read modulo p), and that σ has order p.

   (c) Let $H = H_0$ denote a Sylow p-subgroup of S_p; observe that it is cyclic of order p. Viewing H_0 also a subgroup of S_{p^2}, let H_1 be the image of H_0 under the action of σ; so for example the cycle $(1\,2\,3 \dots p)$ would go to $(p+1\ p+2\ \dots 2p)$. Now let $H_{i+1} = \sigma(H_i)$. Show that S_{p^2} contains a subgroup $K = H_0 \cdots H_{p-1} \cong H_0 \times \cdots \times H_{p-1} \cong H^p$. *Hint: Disjoint permutations commute.*

   (d) Show that $K \cap \langle\sigma\rangle = \{e\}$.

   (e) Now the tricky bit. Rotman wants you to show first that conjugating by σ maps from $H_0 \times \cdots \times H_{p-1}$ to $H_1 \times H_2 \times \cdots \times H_{p-1} \times H_0$. That part is easy. Then he wants you to show that the induced action on $H^p = H_0^p$ is the one you need to identify $\langle K, \sigma\rangle$ as a wreath product. The subtlety is that $H^p \cong H_0 \times \cdots \times H_{p-1}$ and you need to chase through the identifications to verify his claim.

   (f) Conclude that $\langle K, \sigma\rangle \cong H \wr \langle\sigma\rangle \cong \mathbb{Z}_p \wr \mathbb{Z}_p$