Mathematics 101

Fall 2014

Homework 7

The first couple of problems regard towers of groups. By a *normal* tower of groups we mean a chain of subgroups

$$G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_r,$$

that is, where each group G_k is normal in its successor, though not necessarily normal in G_r.

A group G is *solvable* if it admits an *abelian* tower ending in $\{e\}$, that is if there exist subgroups $G_k \le G$ so that

$$\{e\} = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_r = G,$$

where each quotient G_k/G_{k-1} is an abelian group, $1 \le k \le r$.

A *composition series* for a group G (they always exist for finite groups) is a normal tower

$$\{e\} = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_r = G,$$

where each quotient G_k/G_{k-1} is a simple group, $1 \le k \le r$, where simple means having no non-trivial normal subgroups.

1. Let G be a group, and let G' be the *commutator subgroup* of G, the group generated by the set $\{xyx^{-1}y^{-1} \mid x, y \in G\}$.

   (a) Show that if $H \le G$, then $H \supseteq G'$ if and only if $H \trianglelefteq G$, and G/H is an abelian group. In particular $G' \trianglelefteq G$ is the smallest normal subgroup so that G/G' is abelian.

   (b) Show that any group homomorphism $\varphi : G \to H$ where H is abelian factors through the quotient G/G'.

   (c) Given a group G, define a sequence of subgroups $G^{(k)}$ by $G^{(1)} = G'$, the commutator subgroup, and $G^{(k+1)} = [G^{(k)}]'$, the commutator of $G^{(k)}$. Show that G is solvable if and only if $G^{(k)} = \{e\}$ for some $k \ge 1$.

2. Show that the following three statements concerning finite groups are equivalent. The second is the famous Feit-Thompson theorem.

   (a) A finite non-abelian simple group has even order.

   (b) A simple group of odd order is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ for some prime p.

   (c) Every group of odd order is solvable.

3. Let G be a finite group and $H \trianglelefteq G$. Show that G has a composition series one of whose terms is H.

(continued)

4. Group Actions.

   (a) Let G be a finite group, and H a proper subgroup. Show that G is not the union of conjugates of H.

   (b) Let $G = GL_n(\mathbb{C})$, and H the subgroup of lower triangular matrices. Show that G is the union of conjugates of H.

   (c) Let $G = GL_n(\mathbb{R})$, and H the subgroup of lower triangular matrices. Determine whether or not G is the union of conjugates of H (proof or counterexample).

5. Let G be a non-abelian group of order p^3, p a prime. Show that the commutator of G equals its center.

6. Let $p < q$ be primes, and G a non-abelian group of order pq. Show that there is an embedding of G into the symmetric group S_q.