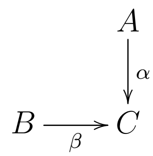
Mathematics 101

Fall 2014

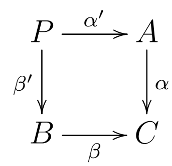
Homework 3

1. (Pullbacks) Given a ring R with identity and R-modules A, B, C, consider the following diagram with R-linear maps α, β:

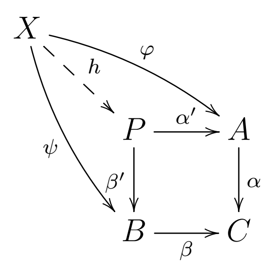


A *pullback* for this diagram (also called a fiber product of α and β) consists of the following data:

- An R-module P and R-linear maps $\alpha' : P \to A$, $\beta' : P \to B$ making the following diagram commute ($\alpha \circ \alpha' = \beta \circ \beta'$).



- It satisfies the following universal mapping property: For every R-module X admitting a commutative diagram of linear maps ($\alpha \circ \varphi = \beta \circ \psi$), there is a unique R-linear map $h : X \to P$ such that the following diagram commutes (that is, $\psi = \beta' \circ h$ and $\varphi = \alpha' \circ h$):



Pullbacks appear frequently in topology where one constructs a pullback of a fiber bundle. Now to the exercise: Let $P = \{(a, b) \in A \times B \mid \alpha(a) = \beta(b)\}$, α' and β' the standard projections to the factors A and B. Show that P together with the associated data form a pullback, i.e., verify that P is an R-module and that the above universal mapping property holds for this choice of P and maps α', β'.

Remark: Dummit and Foote (exercise 27, 10.5) introduce the pullback as a subset of the direct sum $A \oplus B$. Categorically, you should find the direct product a more useful perspective. And of course note that the pullback is unique up to isomorphism by virtue of its definition by universal mapping property.

2. Verify the following are examples of pullbacks.

   (a) With α' and β' the obvious projections:

$$\begin{array}{ccc} A \times B & \xrightarrow{\alpha'} & A \\ {\scriptstyle \beta'}\downarrow & & \downarrow{\scriptstyle \alpha} \\ B & \xrightarrow[\beta]{} & 0 \end{array}$$

   (b) For $\varphi : B \to C$ an R-linear map,

$$\begin{array}{ccc} Ker(\varphi) & \longrightarrow & 0 \\ {\scriptstyle \subseteq}\downarrow & & \downarrow \\ B & \xrightarrow[\varphi]{} & C \end{array}$$

   (c) For $\varphi : B \to C$ an R-linear map and A a submodule of C,

$$\begin{array}{ccc} \varphi^{-1}(A) & \longrightarrow & A \\ {\scriptstyle \subseteq}\downarrow & & \downarrow{\scriptstyle \subseteq} \\ B & \xrightarrow[\varphi]{} & C \end{array}$$

3. Let

$$\begin{array}{ccc} P & \xrightarrow{\alpha'} & A \\ {\scriptstyle \beta'}\downarrow & & \downarrow{\scriptstyle \alpha} \\ B & \xrightarrow[\beta]{} & C \end{array} \quad \text{be a pullback of} \quad \begin{array}{ccc} & & A \\ & & \downarrow{\scriptstyle \alpha} \\ B & \xrightarrow[\beta]{} & C \end{array}$$

   (a) Show that if α is injective, so is β'.

   (b) Show that if α is surjective, so is β'.

4. In the commutative diagram,

$$\begin{array}{ccccc} Q & \xrightarrow{\gamma'} & P & \xrightarrow{\beta'} & A \\ {\scriptstyle \alpha''}\downarrow & & {\scriptstyle \alpha'}\downarrow & & \downarrow{\scriptstyle \alpha} \\ C & \xrightarrow[\gamma]{} & B & \xrightarrow[\beta]{} & D \end{array}$$

   Show that if $\alpha \circ \beta' = \beta \circ \alpha'$ and $\alpha' \circ \gamma' = \gamma \circ \alpha''$ are pullbacks, show that $\alpha \circ (\beta' \circ \gamma') = (\beta \circ \gamma) \circ \alpha''$ is a pullback.

5. (Schanuel's lemma) Let R be a ring, and consider two exact sequences of R-modules

$$0 \longrightarrow K \longrightarrow P \xrightarrow{\varphi} M \longrightarrow 0 \qquad 0 \longrightarrow K' \longrightarrow P' \xrightarrow{\varphi'} M \longrightarrow 0$$

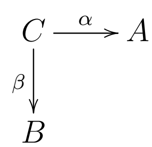
where P and P' are projective. Show that as R-modules $K' \oplus P \cong K \oplus P'$.

Hint: Show there is an exact sequence

$$0 \longrightarrow \ker \pi \longrightarrow X \xrightarrow{\pi} P \longrightarrow 0$$

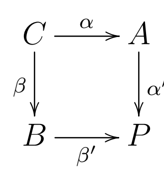
with $\ker \pi \cong K'$ and where X is the fiber product of φ and φ' as in the first problem. From this deduce that $X \cong K' \oplus P$. Similarly, show $X \cong K \oplus P'$.

6. (Pushouts) Given a ring R with identity and R-modules A, B, C, consider the following diagram with R-linear maps α, β:

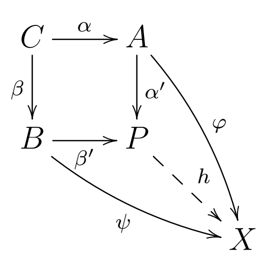


A *pushout* for this diagram consists of the following data:

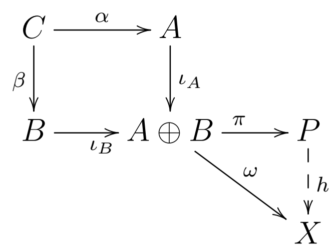
- An R-module P and R-linear maps $\alpha' : A \to P$, $\beta' : B \to P$ making the following diagram commute ($\alpha' \circ \alpha = \beta' \circ \beta$).



- It satisfies the following universal mapping property: For every R-module X admitting a commutative diagram of linear maps ($\varphi \circ \alpha = \psi \circ \beta$), there is a unique R-linear map $h : P \to X$ such that the following diagram commutes (that is, $\psi = h \circ \beta'$ and $\varphi = h \circ \alpha'$):



Define the pushout as follows: Using the universal mapping property of the coproduct, each pair of homomorphisms $\varphi : A \to X$ and $\psi : B \to X$ can be described by a single homomorphism $\omega : A \oplus B \to X$. Let $\iota_A : A \to A \oplus B$ and $\iota_B : B \to A \oplus B$ be the standard injections. Put $P = (A \oplus B)/K$ where $K = Im(\iota_A \circ \alpha - \iota_B \circ \beta) = \{(\alpha(c), -\beta(c)) \in A \oplus B \mid c \in C\}$ and let $\pi : A \oplus B \to P$ be the quotient map. Put $\alpha' = \pi \circ \iota_A$ and $\beta' = \pi \circ \iota_B$. Show that $\alpha' \circ \alpha = \beta' \circ \beta$ is a pushout.



Remark: You will use the notion of a pushout in the proof of the Siefert-van Kampen theorem.