Math 101 Fall 2013
Homework #5
Due Wednesday October 30, 2013

1. Show that if $G/Z(G)$ is cyclic, then G is abelian. (This completes our characterization of groups of order p^2 from lecture.)

2. Let G be the alternating group A_4 on four letters.

(a) Show that if G has a subgroup of order 6, then that subgroup would be normal.

(b) Conclude that if H is a subgroup of order 6, then H contains every element of order 3.

(c) Notice that A_4 has at least 8 elements of order 3.

(d) Conclude that A_4 has no subgroup of order 6 even though $6 \mid |A_4|$. Hence the converse of Lagrange's Theorem is not true.

3. Let $D_8 = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$ be the dihedral group (of symmetries of the square) so that $rs = sr^{-1}$. Observe that

$$\langle s \rangle \lhd \langle s, r^2 \rangle \lhd D_8,$$

but $\langle s \rangle \ntriangleleft D_8$.

4. Suppose that $Z(G)$ has index n in G. Then prove that every conjugacy class has at most n elements.

5. Prove that if $n \geq 3$, then $Z(S_n) = \{1\}$.

6. Let $|A| > 1$ and let G be a subgroup of S_A that acts transitively on A. Show that there is a $\sigma \in G$ such that $\sigma(a) \neq a$ for all $a \in A$. (One says σ is fixed point free.)