Math 101 Fall 2013
Homework #5
Due Wednesday October 30, 2013

1. Show that if $G/Z(G)$ is cyclic, then G is abelian. (This completes our characterization of groups of order p^2 from lecture.)

ANS: Let $q : G \to G/Z(G)$ be the quotient map and let $q(x)$ be a generator for $G/Z(G)$. Let y and x be elements of G. Then $q(y) = q(x)^n$ and $q(z) = q(x)^m$. It follows that $y = x^n a$ and $z = x^m b$ for $a, b \in Z(G)$. But then

$$yz = x^n a y^m b = x^{n+m} ab = x^m x^n ba = x^m b x^n a = zy.$$

Since x and y were arbitrary, G is commutative.

2. Let G be the alternating group A_4 on four letters.

(a) Show that if G has a subgroup of order 6, then that subgroup would be normal.

(b) Conclude that if H is a subgroup of order 6, then H contains every element of order 3.

(c) Notice that A_4 has at least 8 elements of order 3.

(d) Conclude that A_4 has no subgroup of order 6 even though $6 \mid |A_4|$. Hence the converse of Lagrange's Theorem is not true.

ANS: (a) If $|H| = 6$, then $[G : H] = 2$ and $H \lhd G$.

(b) With H as above, G/H is $\mathbf{Z}_2$ and $a^2 \in H$ for all $a \in G$. But if $|x| = 3$, then $x = x^4 = (x^2)^2 \in H$.

(c) As counted in class, S_4 has eight distinct 3-cycles, and of course, each of these has order 3 and is even. Hence A_4 has eight elements of order 3.

(d) Since any subgroup of order 6 in A_4 would have to have at least eight elements ....

3. Let $D_8 = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$ be the dihedral group (of symmetries of the square) so that $rs = sr^{-1}$. Observe that

$$\langle s \rangle \lhd \langle s, r^2 \rangle \lhd D_8,$$

but $\langle s \rangle \ntriangleleft D_8$.

ANS: Since $r^2 s = sr^{-2} = sr^2$, note that $\langle s, r^2 \rangle = \{1, r^2, s, sr^2\}$. Hence $|\langle s, r^2 \rangle| = 4$. Thus both subgroups have index 2 and must be normal. But $rsr^{-1} = sr^{-2} = sr^2$, so $\langle s \rangle$ is not normal in D_8.
Remark: Note that $\langle s \rangle$ can't be characteristic in $\langle s, r^2 \rangle$.

4. Suppose that $Z(G)$ has index n in G. Then prove that every conjugacy class has at most n elements.

ANS: Let $x \in G$ and let O_x be its conjugacy class. We have

$$|O_x| = [G : C_G(x)].$$

But $Z(G) < C_G(x)$ and $n = [G : Z(G)] = [G : C_G(x)][C_G(x) : Z(G)]$. Hence $|O_x| \leq n$ as claimed.

5. Prove that if $n \geq 3$, then $Z(S_n) = \{1\}$.

ANS: Let $\sigma \in S_n \setminus \{1\}$. Say $\sigma(i) = j \neq i$. Since $n \geq 3$, we can find k be different from both i and j. Then $\tau = (j,k)\sigma(j,k)$ is comjugate to σ and $\tau(i) = k \neq j$ so $\tau \neq \sigma$. Thus $\sigma \notin Z(G)$. Hence $Z(G) = \{1\}$.

6. Let $|A| > 1$ and let G be a subgroup of S_A that acts transitively on A. Show that there is a $\sigma \in G$ such that $\sigma(a) \neq a$ for all $a \in A$. (One says σ is fixed point free.)

ANS: Let $a \in A$ and let $G_a = \{\, \sigma \in G : \sigma(a) = a \,\}$. If $G \cdot a$ is the orbit of a, then $|G \cdot a| = [G, G_a] = |G|/|G_a|$. Since $G \cdot a = A$, we see that $|G_a| = |G|/|A|$. Since $1 \in G_a$ for all a, we have

$$\Big| \bigcup_{a \in A} G_a \Big| < \sum_{a \in A} |G_a| = |A| \cdot \frac{|G|}{|A|} = |G|.$$

Thus there is a $\sigma \in G$ such that $\sigma \notin \bigcup_a G_a$. But then

$$\sigma \in \bigcap_{a \in A} G_a^c = \{\, \sigma \in G : \sigma(a) \neq a \text{ for all } a \in A \,\}.$$

That is, σ is fixed point free.