Math 101 Fall 2013
First Homework
Due Wednesday September 25, 2013

1. Recall that if k is a field and $\beta = \{ v_1, \dots, v_n \}$ is a basis for a k-vector space V, then there is a vector space isomorphism $\Phi : V \to k^n$ given by sending $v \in V$ to its coordinate vector $[v]_\beta = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ where the c_i are the unique scalars such that $v = c_1v_1 + \cdots + c_nv_n$. If W is another k-vector space with basis $\alpha = \{ w_1, \dots, w_m \}$ and if $T : V \to W$ is a linear transformation, then by definition $[T]_\beta^\alpha$ is the $m \times n$ matrix whose j^{th} column is $[Tv_j]_\alpha$. Recall that if $A = (a_{ij})$ is a $m \times n$ matrix and $B = (b_{ij})$ is a $n \times p$ matrix then AB is the $m \times p$ matrix (c_{ij}) with $c_{ij} = \sum_k a_{ik}b_{kj}$. You may want to use the observation (after having checked it without including it in your homework write-up) that the j^{th} column of AB is Ac where c is the j^{th} column of B.

(a) Let V, W, β, α and T be as above. Show that

$$[Tv] = [T]_\beta^\alpha [v]_\beta.$$

(b) Suppose that $\gamma = \{ z_1, \dots, z_p \}$ is a basis for a k-vector space Z and that $S : W \to Z$ is linear. Show that

$$[ST]_\beta^\gamma = [S]_\alpha^\gamma [T]_\beta^\alpha.$$

(c) Let $F : \mathbf{R}^2 \to \mathbf{R}^2$ be reflection across the line $y = (\tan\theta)x$. Let $\sigma = \{ e_1, e_2 \}$ be the standard basis for $\mathbf{R}^2$. Find $[F]_\sigma^\sigma$. (I suggest the following. Let $u = (\cos\theta, \sin\theta)$ and $w = (-\sin\theta, \cos\theta)$. Then $\beta = \{ u, w \}$ is a basis for $\mathbf{R}^2$ and since $F(u) = u$ and $F(w) = -w$, the matrix $[F]_\beta^\beta$ has a particularly simple form. But by part (b) above,

$$[F]_\sigma^\sigma = [I]_\beta^\sigma [F]_\beta^\beta [I]_\sigma^\beta$$

where $I : \mathbf{R}^2 \to \mathbf{R}^2$ is the identity map. However one of $[I]_\beta^\sigma$ and $[I]_\sigma^\beta$ is easy to compute and the other is its inverse. For your final answer, you should employ the sum formulas for sin and cos.)

2. Let $\{X_\alpha\}_{\alpha\in A}$ be a collection of sets (a.k.a. a "set of sets", which just sounds awful to me). Let

$$C := \{\, (\alpha, x) \in A \times \bigcup_{\alpha\in A} X_\alpha : x \in X_\alpha \,\}.$$

(Because we can identify X_α with $\{\, (\alpha, x) : x \in X_\alpha \,\}$, C is sometimes called the *disjoint union* of the X_α. For example, think about the case where the X_α are all the same. Then C is quite different from the union.) Show that in the category of sets and functions, the coproducts exist and are given by the disjoint union.

3. Let $\mathscr{C}$ be a category in which products and coproducts exist. Recall that $\hom_{\mathscr{C}}(X, Y)$ is a set for any pair of objects in $\mathscr{C}$. Show that there is a unique isomorphism

$$\phi : \hom_{\mathscr{C}}\Big(Y, \prod_{\alpha\in A} X_\alpha\Big) \to \prod_{\alpha\in A} \hom_{\mathscr{C}}(Y, X_\alpha)$$

such that $\pi_\alpha \circ \phi(h) = p_\alpha \circ h$. (Here π_α and p_α are the natural projections for the product in category of sets and maps, and for the product in $\mathscr{C}$, respectively.)

Similarly, show that there is a unique isomorphism

$$\psi : \hom_{\mathscr{C}}\Big(\coprod_{\alpha\in A} X_\alpha, Y\Big) \to \prod_{\alpha\in A} \hom_{\mathscr{C}}(X_\alpha, Y)$$

such that $\pi_\alpha \circ \psi(h) = h \circ i_\alpha$.

4. Note that in the category of R-modules, we can think of $\bigoplus_{i=1}^n M_i$ as either the product or the coproduct of the finite set $\{\, M_1, \dots, M_n \,\}$. Let $\kappa_k : M_k \to \bigoplus_{i=1}^n M_i$ and $\pi_k : \bigoplus_{i=1}^n M_i \to M_k$ be the natural maps. In this instance, question 3 says we can identify the set $\hom\Big(\bigoplus_{i=1}^n M_i, \bigoplus_{j-1}^r N_j\Big)$ with the set $\bigoplus_{i=1,j=1}^{n,r} \hom(M_i, N_j)$; specifically, we identify h with the matrix $[h] = (h_{ij})$ where $h_{ij} = \pi_i \circ h \circ \kappa_j \in \hom(M_j, N_i)$. Thus

$$h(m_1, \dots, m_n) = \Big(\sum h_{1j}(m_j), \sum h_{2j}(m_j), \dots, \sum h_{rj}(m_j)\Big)$$

Verify that if $h \in \hom\Big(\bigoplus_{i=1}^n M_i, \bigoplus_{j-1}^r N_j\Big)$ and $k \in \hom\Big(\bigoplus_{j-1}^r N_j, \bigoplus_{k=1}^s P_k\Big)$, then $[k \circ h] = [k][h]$ (with the obvious interpretation of $[k][h]$).

5. Suppose that V and W are finite-dimensional k-vector spaces over the field k. Let $T : V \to W$ be a linear map. Show that there are bases β of V and α of W such that $[T]_\beta^\alpha$ is diagonal (i.e., all off-diagonal entries zero) with diagonal entries in $\{0, 1\}$. (I used the proof of the rank-nullity theorem as a guide.)

6. Suppose that V is a finite-dimensional k-vector space and that $T : V \to V$ is linear. Show that V has a basis β such that $[T]_\beta^\beta$ is diagonal with entries in $\{0, 1\}$ (as in question 5) if and only if $T = T^2$. Compare with the result from question 5.

7. Let V and W be k-vector spaces as above. Then $\hom_k(V, W)$ is just a fancy way of describing the set of linear maps from V to W. After picking a bases for V and W, we can identify $\hom_k(V, W)$ with the set $M_{m\times n}(k)$ of $m \times n$ matrices where $m = \dim W$ and $n = \dim V$. We write $\mathrm{GL}_m(k)$ for the invertible $m \times m$-matrices. Recall that A and B in $M_{m\times n}(k)$ are row-equivalent if and only if there is a $P \in \mathrm{GL}_m(k)$ such that $PA = B$ and that each such A is row-equivalent to a unique matrix R is row-reduced echelon form.

(a) Define an equivalence relation on $\hom_k(V, W)$ so that $T \sim S$ if and only if there is an isomorphism $U : W \to W$ such that $S = UT$. If k is a finite field with p elements, $\dim V = 4$ and $\dim W = 2$, then how may equivalence classes are there?

(b) Now define $T \approx S$ if there are isomorphisms $U_1 : V \to V$ and $U_2 : W \to W$ so that $S = U_2 T U_1$. How many $\approx$-equivalence classes are there if $\dim V = n$ and $\dim W = m$?