Math 101 Fall 2013
Final Exam
Due Monday, November 25, 2013

INSTRUCTIONS: You are allowed to use your lecture notes and a textbook of your choice (either Lang or one of the other texts on reserve). No other resources are allowed — animate or inanimate — with the one exception that you can ask me for clarification. Monitor the web page for corrections and typos.

The exam is due by Monday, November 25, 2013 at 11am. You can email me a PDF if you use LaTeX. Otherwise, please had the exam to me personally, or at the very least, make a xerox copy. If you are not using LaTeX, then use one side of the paper only and start each problem on a separate page.

Unless stated otherwise, R denotes a (possibly noncommutative) ring with identity. The identity of a subring is always the identity of the parent ring. Ideal always means two-sided ideal.

1. (10) Let $A = \mathbf{Z}_{360} \times \mathbf{Z}_{10}$ and $B = \mathbf{Z}_{154} \times \mathbf{Z}_2$. Give an invariant factor decomposition for $C = A \otimes_{\mathbf{Z}} B$.

2. (20) Let

$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\phi} N \longrightarrow 0 \qquad (\dagger)$$

be a short exact sequence of left R-modulues.

(a) Show that $(\dagger)$ splits if and only if

$$0 \longrightarrow \hom_R(D, L) \xrightarrow{\psi'} \hom_R(D, M) \xrightarrow{\phi'} \hom_R(D, N) \longrightarrow 0$$

is exact for all R-modules D.

(b) Show that $(\dagger)$ splits if and only if

$$0 \longrightarrow \hom_R(N, D) \xrightarrow{\phi'} \hom_R(M, D) \xrightarrow{\psi'} \hom_R(L, D) \longrightarrow 0$$

is exact for all R-modules D.

(For the "if" part of part (a), consider $D = N$ and the identity map in $\hom_R(N, N)$.)

3. (10) Suppose that R is a subring of the commutative ring E. Notice that $E \otimes_R R[x]$ is an E-algebra with respect to the multiplication $(s \otimes f(x))(s' \otimes g(x)) = ss' \otimes f(x)g(x)$ and the obvious E-module structure. Show that $E \otimes_R R[x]$ and $E[x]$ are isomorphic as E-*algebras*.

4. (10) Let V be a vector space over $\mathbf{Q}$ and let T be an invertible operator on V such that $T^{-1} = T^2 + T$. Show that $\dim V$ is a multiple of 3. Find a representative of each similarity class of such operators if $\dim V = 6$.

5. (10) Let I be a principal ideal in an integral domain R. Prove that the R-module $I \otimes_R I$ is torsion free. That is, show that if $r \cdot m = 0$ with $r \in R$ and $m \in I \otimes_R I$, then $m = 0$.

6. (20) If M is a left R-module, then we can make the abelian group $\hom_R(M, R)$ into a *right* R-module via $\phi \cdot r(m) := \phi(m)r$. The right module $\hom_R(M, R)$ is called the *dual module* to M.

(a) Show that $\hom_R(R, R) \cong R$ (as right R-modules).

(b) Show if F is a finitely generated free left R-module, then $\hom_R(F, R)$ is a finitely generate free right R-module of the same rank.

(c) Show that if M is a finitely generated projective left R-module, then $\hom_R(M, R)$ is projective.

(d) If $\phi \in \hom_R(M, R)$ and N is a right R-module, show that there is a map $s_\phi : N \otimes_R M \to N$ such that $s_\phi(n \otimes m) = n \cdot \phi(m)$.

7. (20) Suppose that G is a group of order $693 = 3^2 \cdot 7 \cdot 11$ with a *normal* Sylow 3-subgroup P.

(a) Show that G has a unique Sylow 11-subgroup R.

(b) If $a \in P$, show that $R \subset C_G(a)$. (Hint: consider $P \cap R$.)

(c) If $Q \in \mathrm{Syl}_7(G)$ and $a \in P$, show that $Q \cap C_G(a) \neq \{1\}$. (Hint: Let Q act on P by conjugation.)

(d) Conclude that $P \subset Z(G)$.

(e) Conclude that G is abelian.