Dartmouth College

Mathematics 101

Homework 6 (due Thursday, Nov 12)

1. In a ring R without identity, one can still define the characteristic as zero or the least positive integer n such that $na = 0_R$ for all $a \in R$.

   Show that every ring R without identity can be embedded in a ring S with identity in at least two ways; one in which S has characteristic zero and the other where S has the same characteristic as R.

   For the characteristic zero model, let $S = R \oplus \mathbb{Z}$ be the additive abelian group, and define a multiplication by $(r_1, k_1)(r_2, k_2) = (r_1 r_2 + k_2 r_1 + k_1 r_2, k_1 k_2)$. Verify that S is a ring with identity $(0, 1)$ and characteristic 0, and that the map $R \to S$ given by $r \mapsto (r, 0)$ is an injective ring homomorphism.

   When the characteristic of R is $n > 0$, do a similar construction with $S = R \oplus \mathbb{Z}/n\mathbb{Z}$.

2. Let R be a commutative ring with identity, and let $\mathfrak{N}(R) = \{x \in R \mid x \text{ is nilpotent}\}$; nilpotent means that $x^n = 0$ for some positive integer n.

   (a) $\mathfrak{N}(R)$ is called the nilradical of R; show that $\mathfrak{N}(R)$ is an ideal of R.

   (b) Show that $\mathfrak{N}(R) = \cap_{\mathfrak{P}} \mathfrak{P}$ where $\mathfrak{P}$ ranges over all the prime ideals of R. Hint: The inclusion $\mathfrak{N}(R) \subseteq \cap_{\mathfrak{P}} \mathfrak{P}$ is quite straightforward. For the other consider the contrapositive. Let $x \in R$ which is not nilpotent, and let S be the set of ideals I in R with the property that $x^n \notin I$ for $n \geq 1$. Show that Zorn's lemma applies to S and that a maximal element is a prime ideal not containing the element x.

3. If $\varphi : R \to S$ is a surjective homomorphism of rings, show that there is a one-to-one correspondence between the prime ideals of S and the prime ideals of R which contain $ker(\varphi)$.

4. Characterize the prime ideals of $R = \mathbb{Z}/n\mathbb{Z}$ ($n > 1$), and determine its nilradical, $\mathfrak{N}(R)$.

5. For R a commutative ring with identity, and $I \subseteq R$ an ideal, define the *radical* of I, $\text{rad}(I) = \sqrt{I}$, to be $\{r \in R \mid r^n \in I \text{ for some } n \geq 1\}$. An ideal I is called a *radical ideal* if $\sqrt{I} = I$.

   (a) Show that $\sqrt{I}$ is an ideal containing I

(b) Show that every prime ideal is radical.

(c) Show that $\sqrt{I}/I = \mathfrak{N}(R/I)$.

(d) Show that $\sqrt{I}$ is the intersection of all prime ideals of R which contain I.

6. The Jacobson radical of a ring R with identity is the intersection of all the maximal (left) ideals of R. More generally, we can define the Jacobson radical of an ideal I, $\mathrm{Jac}(I)$, to be the intersection of all maximal left ideals which contain I; then the Jacobson radical of R is simply $\mathrm{Jac}(0)$. Below we assume R is commutative.

   (a) Show that the Jacobson radical of R contains the nilradical.

   (b) Show that for an ideal I, $\sqrt{I} \subset Jac(I)$.

   (c) Compute the Jacobson radical of $\mathbb{Z}/n\mathbb{Z}$, $n \geq 2$.