Dartmouth College

Mathematics 101

Homework 2 (due Thursday, Oct 8)

1. Let G be a group and $H \leq G$. Recall that the normalizer of H in G is defined by $N_G(H) = \{g \in G \mid gHg^{-1} = H\}$. Of course if H is a finite group, this condition can be weakened to $gHg^{-1} \subseteq H$, however this is not true in general. In particular let $G = GL_2(\mathbb{Q})$ and H the following group of unipotent matrices in $SL_2(\mathbb{Z})$: $H = \{\left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right) \mid m \in \mathbb{Z}\}$.

   (a) Characterize the diagonal matrices $g \in GL_2(\mathbb{Q}) \cap M_2(\mathbb{Z})$ with $gHg^{-1} \subseteq H$.

   (b) Show that there exist g among those just found which do not live in the normalizer.

2. Let G be a cyclic group of order n, and let $d \in \mathbb{Z}_+$ with $d \mid n$. Show that G has exactly d elements of exponent d.

3. Let G be a group with subgroups, H, K each of finite index in G, say $[G : H] = m$ and $[G : K] = n$. Show that $H \cap K$ has finite index in G. More specifically show that

$$\text{lcm}\{m, n\} \leq [G : H \cap K] \leq mn.$$

4. For a group G, the *center* of G, denoted Z_G is defined by:
   $Z_G = \{x \in G \mid xg = gx \text{ for all } g \in G\}$. Show that if G/Z_G is cyclic, then G is abelian.

5. Let $Aut(G)$ denote the group of automorphisms of a group G, and $Inn(G)$ the subgroup of inner automorphisms. That is, $Inn(G) = \{\varphi_g \mid g \in G\}$ where $\varphi_g : G \to G$ is defined by $\varphi_g(x) = gxg^{-1}$.

   (a) Show that $Inn(G) \trianglelefteq Aut(G)$.

   (b) Show that $G/Z_G \cong Inn(G)$.

   (c) Show that if $Aut(G)$ is cyclic, then G is abelian.

6. Let G be a finite group, and assume N is a normal subgroup of G with
   $\gcd(|N|, [G : N]) = 1$. Show that N is the only subgroup of G of order $|N|$.