Dartmouth College
Mathematics 101

Assignment 1 (aka, Awakening from hibernation):
due Thursday, October 1

It is assumed you have read Chapter 0 and Chapter 1.1 - 1.6 as well as the few sections we are covering for this week in Chapter 2. The following set of exercises is meant to reengage your brains by having you play with some explicit examples and by helping you recall some basic facts from your undergraduate linear and abstract algebra courses.

1. Let $\sigma = (a_1 a_2 \ldots a_m)$ be an m-cycle in the symmetric group S_n $(n \geq m)$.
   (a) (p33, #10) Show that for $1 \leq i, k \leq m$, $\sigma^i(a_k) = a_{k+i}$ where the subscript $k+i$ is interpreted as the least positive residue of $k+i$ modulo m. Deduce that the order of σ is m.
   (b) (p33, #11 modified) Show that σ^i is an m-cycle if and only if $\gcd(i, m) = 1$.
2. (p33, #15) Prove that the order of an element in S_n equals the least common multiple of the lengths of its cycles in its cycle decomposition. Hint: See also exercise 24, in 1.1.
3. (p35) Establish the formula at the end of section 1.4: Let $\mathbb{F}_q$ be a finite field with q elements. Show that $|GL_n(\mathbb{F}_q)| = (q^n - 1)(q^n - q) \cdots (q^n - q^{n-1})$. Hint: Recall that a matrix is invertible if and only if its rows (resp. columns) are linearly independent.
4. (p40, #4) Show that the multiplicative groups $\mathbb{R}^\times$ and $\mathbb{C}^\times$ are not isomorphic.
5. (p41, #25) Let n be a positive integer and let r, s be the canonical generators of the dihedral group D_{2n}, and let $\theta = 2\pi/n$.
   (a) Prove that the matrix $\left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right)$ is the matrix of the linear transformation (with respect to the standard basis of $\mathbb{R}^2$) which describes a counterclockwise rotation about the origin of θ radians.
   (b) Prove that the map $\varphi : D_{2n} \to GL_2(\mathbb{R})$ defined on generators by $\varphi(r) = \left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right)$ and $\varphi(s) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ extends to an injective group homomorphism.
6. Let $G = GL_2(F)$ for a field F. Let H be the subgroup of upper triangular matrices in G (you may assume H is a group).
   (a) If $F = \mathbb{C}$, show that $G = \bigcup_{g \in G} gHg^{-1}$, that G is a union of conjugates of H. Hint: This is really a linear algebra problem. Let $T : \mathbb{C}^2 \to \mathbb{C}^2$ be any linear transformation. Show that there exists a basis $\mathcal{B}$ for $\mathbb{C}^2$ so that the matrix of T with respect to $\mathcal{B}$ is upper triangular. Hint for hint: think about eigenvectors.
   (b) If $F = \mathbb{R}$, does the result remain valid? Proof or counterexample.