Dartmouth College

Mathematics 101

Homework 7 (due Wednesday, November 28)

1. Let A be a commutative ring with identity. Let $I_1, \ldots, I_n$ be ideals in A which are coprime in pairs. Show that $I_1 I_2 \cdots I_n = I_1 \cap I_2 \cap \cdots \cap I_n$.

2. Let A be a commutative ring with identity, and let X be the set of all prime ideals in A. X is called the prime spectrum of A, written $Spec(A)$. For each subset $E \subseteq A$, let $V(E)$ denote the set of primes ideals of A which contain E. Prove that:

   (a) If $I = \langle E \rangle$ is the ideal generated by E, then $V(I) = V(E)$.

   (b) Show that $V(0) = X$ and $V(1) = \emptyset$.

   (c) If $\{E_i\}_{i \in I}$ is any family of subsets of A, then $V(\cup_i E_i) = \cap_{i \in I} V(E_i)$.

   (d) For any ideals I, J of A, show that $V(I \cap J) = V(IJ) = V(I) \cup V(J)$.

   These properties demonstrate that the sets $V(E)$ satisfy the axioms for closed sets in a topological space. This topology is called the Zariski topology on $Spec(A)$.

3. Let $A = \mathbb{Z}$ and $\mathfrak{p} = p\mathbb{Z}$ with p a prime in $\mathbb{Z}$. We have characterized the localization $A_{\mathfrak{p}} = \mathbb{Z}_{\mathfrak{p}}$ as $\{a/b \in \mathbb{Q} \mid a, b \in Z,\ p \nmid b,\ \gcd(a, b) = 1\}$.

   (a) Characterize the unit group $\mathbb{Z}_{\mathfrak{p}}^{\times}$.

   (b) Show that every nonzero element in $\mathbb{Z}_{\mathfrak{p}}$ can be written uniquely as $p^{\nu} u$ where ν is a nonnegative integer and $u \in \mathbb{Z}_{\mathfrak{p}}^{\times}$. You may of course assume unique factorization in $\mathbb{Z}$.

   (c) Characterize all the ideals of $\mathbb{Z}_{\mathfrak{p}}$, and confirm that $\mathbb{Z}_{\mathfrak{p}}$ has a unique maximal ideal.

   (d) Show that $\mathbb{Z}_{\mathfrak{p}} / p\mathbb{Z}_{\mathfrak{p}} \cong \mathbb{Z}/p\mathbb{Z}$.

4. Consider the localization of $\mathbb{Z}[x]$ at the prime ideal (x).

   (a) Describe the elements of $\mathbb{Z}[x]_{(x)}$.

   (b) Is (x) maximal in $\mathbb{Z}[x]_{(x)}$? If so, describe the resulting quotient field.

   (c) How does $\mathbb{Z}[x]_{(x)}$ compare to $\mathbb{Q}[x]_{(x)}$?