Dartmouth College

Mathematics 101

Homework 4 (due Wednesday, Oct 24)

1. Show that the following three statements are equivalent. The second is the Feit-Thompson theorem.

   (a) A finite non-abelian simple group has even order.

   (b) A simple group of odd order is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ where p is a prime.

   (c) Every group of odd order is solvable.

2. Let G be a finite group and H a proper subgroup. Show that G is not the union of conjugates of H.

3. Let G be a finite simple group containing an element of order 21. Show that every proper subgroup H of G satisfies $[G : H] \geq 7$ and show $|G| \geq 168$.

4. Let p be a prime, and G a p-group, i.e., $|G| = p^m$ for some $m \geq 1$.

   (a) Suppose that H is a nontrivial normal subgroup of G. Show that $H \cap Z_G \neq \{e\}$ where Z_G is the center of G. In particular, every normal subgroup H of order p is contained in the center.

   (b) If G is nonabelian of order p^3, show that $Z_G = G'$ (the commutator), and they are both cyclic of order p.