Dartmouth College

Mathematics 101

Homework 3 (due Wednesday, Oct 17)

1. Let G be a finite group, and for positive integers n define:

$$\delta(n) = \#\{x \in G \mid |x| = n\}, \text{ and}$$
$$\varepsilon(n) = \#\{x \in G \mid x^n = e\}.$$

   (a) Show that $\varepsilon(n) = \sum_{m|n} \delta(m)$.

   (b) Show that $\varepsilon(n) \leq n$ implies $\delta(n) = 0$ or $\phi(n)$; here ϕ is the Euler ϕ-function.

   (c) If G is cyclic and $m \mid |G|$, show that $\delta(m) = \phi(m)$ and $\varepsilon(m) = m$.

   (d) Show that G is cyclic if and only if $\varepsilon(n) \leq n$ for all $n \geq 1$. *Hint:* It may be useful to show that $\varepsilon(n) = \varepsilon(d)$, where $d = \gcd(n, |G|)$.

2. Let G be a group and let G' be the subgroup of G generated by the set $\{xyx^{-1}y^{-1} \mid x, y \in G\}$. G' is called the commutator subgroup of G.

   (a) Show that if H is a subgroup of G, then $H \supseteq G'$ if and only if $H \triangleleft G$ and G/H is abelian. In particular, $G' \triangleleft G$ and G/G' is abelian.

   (b) Show that if $\varphi : G \to H$ is a homomorphism of groups, and H is abelian, then φ factors through G/G', that is there is a map $\varphi_* : G/G' \to H$ with $\varphi = \varphi_* \circ \pi$ with $\pi : G \to G/G'$ the standard projection.

3. Solvable groups.

   (a) Let G be a group, and H a normal subgroup. Show that G is solvable if and only if H and G/H are solvable.

   (b) Let G be a group, $H \trianglelefteq G$ with G/H finite and abelian. Show that there is a sequence of subgroups H_i with $H = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_n = G$ with $H_i \trianglelefteq H_{i+1}$ and H_{i+1}/H_i cyclic for $0 \leq i < n$.

   (c) For a group G, define a sequence of subgroups $G^{(k)}$, by $G^{(1)} = G'$ (the commutator subgroup), and $G^{(k+1)} = [G^{(k)}]'$, the commutator of $G^{(k)}$. Show that G is solvable if and only if $G^{(k)} = \{e\}$ for some $k \geq 1$.