Dartmouth College

Mathematics 101

Homework 3 (due Wednesday, October 15)

1. Let Z_G denote the center of the group G. Prove that if G/Z_G is cyclic, then G is abelian.

2. Recall that a group is simple if it has no nontrivial normal subgroups. A normal subgroup N of a group G is *maximal normal* if $N \neq G$ and whenever M is a normal subgroup of G with $N \subseteq M \subseteq G$, then $M = N$ or $M = G$. Show that N is maximal normal if and only if G/N is simple and nontrivial.

3. Let G be a group and let G' be the subgroup of G generated by the set $\{xyx^{-1}y^{-1} \mid x, y \in G\}$. G' is called the commutator subgroup of G.

   (a) Show that if H is a subgroup of G, then $H \supseteq G'$ if and only if $H \triangleleft G$ and G/H is abelian. In particular, $G' \triangleleft G$ and G/G' is abelian.

   (b) Show that if $\varphi : G \to H$ is a homomorphism of groups, and H is abelian, then φ factors through G/G', that is there is a map $\widetilde{\varphi} : G/G' \to H$ with $\varphi = \widetilde{\varphi} \circ \pi$ with $\pi : G \to G/G'$ the standard quotient map.

4. Let N be a normal subgroup of a finite group G such that $\gcd(|N|, [G : N]) = 1$. Show that N is the only subgroup of G with order $|N|$.

5. Let p be the smallest prime dividing the order of a finite group G. Let H be a subgroup of G with index p. Show that $H \triangleleft G$.