Limit complexity of finite and infinite sequences

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FRG workshop, 2007

Outline of the talk

- Limit complexity of infinite computable sequences
- Limit complexity of finite sequences
- Applications to 2-randomness

Complexities of computable sequences

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- $N(\omega) = \max C(\omega_n|n)$

$$N(\omega) \leqslant C(\omega)$$

Theorem (Durand, Shen, V. 1999)

$\exists\omega\ N(\omega) \ll C(\omega)$

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$$\omega = 00\ldots01111111111111\ldots$$

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Theorem (Durand, Shen, V. 1999)

$\exists\omega \; N_\infty(\omega) \leqslant C_\infty(\omega) \ll N(\omega) \leqslant C(\omega)$

Proof of $\exists\omega\ C_\infty(\omega) \ll N(\omega)$

T. Kamae's example (1973): let x be the lex first string of length m and complexity $\geqslant m$. Then

$$\forall^\infty n \quad C(x|n) \ll C(x)$$

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Comparing limit complexities

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- *and, moreover,* $N(\omega) = m$.

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Why it works? [Picture]

Proof of $\exists\omega\ C_\infty(\omega) = 2m,\ N(\omega) = m$

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The non-uniform version of $C_\infty(x)$ is

$$N_\infty(x) = \limsup C(x|n)$$

Comparing limit complexities

Theorem

[authors]

$$C_\infty(x) = C^{0'}(x)$$

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Theorem (V. 2002)

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Theorem (V. 2002)

There is a Σ_2 set of cardinality $\leqslant 2^m$ covering all such x's

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We have $\mu(U) \leqslant \varepsilon$
$\Rightarrow (\forall \varepsilon' > \varepsilon)\ (\exists \text{ open } V \supset U)\ \mu(V) < \varepsilon'$.

Question: Given a uniformly effectively open family U_n is there a $0'$-effectively open such V?

Partial positive answers (LMSV)

- There exists an effectively open covering of measure ε of a smaller set

$$\bigcup_N \operatorname{Int}\Big(\bigcap_{n>N} U_n\Big)$$

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- There exists an effectively open covering of measure ε of a smaller set

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- Yes, if U_n has "effectively bounded granularity".

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Proof of $\Rightarrow$-part:

$d(\omega_n) > m \Leftrightarrow \omega \in U_n$

where $U_n = \bigcup_{|x|=n,\ d(x)>m} \Omega_x$

Relation to 2-randomness (continued)

Another randomness deficiency:

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$\bar{d}(x) \geqslant m \Leftrightarrow (\forall n \geqslant |x|)\ \Omega_x \subset U_n$

where $U_n = \bigcup_{|y|=n,\ d(y)\geqslant m} \Omega_y$

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Theorem (Bienvenue, Muchnik, Shen, V.)

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Theorem

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Theorem (Muchnik 1987)

There is a computable sequence $a_1, a_2, a_3, \ldots$ such that $f(x) = m^{0'}(x)$.

Refernces available on-line

- Laurent Bienvenu, Andrej Muchnik, Alexander Shen, Nikolay Vereshchagin. Limit complexities revisited. http://lpcs.math.msu.su/~ver/papers/laurent.pdf
- B. Durand, A. Shen, and N. Vereshchagin. Descriptive complexity of computable sequences. http://lpcs.math.msu.su/~ver/papers/seq.ps
- N. Vereshchagin. Complexity Conditional to Large Integers. http://lpcs.math.msu.su/~ver/papers/lc.ps