

# NONELEMENTARY ANTIDERIVATIVES

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*This is a preliminary version, and has not been carefully proof-read. Please do not duplicate or circulate. I will be happy to provide a final version on request. In the meantime, I would be delighted to get your comments, suggestions, and especially corrections.*

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ABSTRACT. I will attempt to give a self-contained treatment which will make precise and then prove the statement that “the antiderivative of  $\exp(-x^2)$  can not be written in terms of the usual, so-called elementary, functions of Calculus.

## 1. INTRODUCTION

For years I’ve begun teaching techniques of integration to first year college students by warning them that integration—more precisely antidifferentiation—is harder than differentiation. In particular, I am fond of pointing out that “it is easy to write down simple expressions, such as  $\exp(-x^2)$ , that don’t have an antiderivative which can be expressed in terms of the standard functions of calculus.” Of course, it is a tad difficult to explain precisely what one means. In the previous semester, I was quite happy to point out that by the Fundamental Theorem of Calculus every continuous function had an antiderivative; so

$$\operatorname{erf}(x) = \int_0^x \exp(-t^2) dt$$

is certainly an antiderivative of  $\exp(-x^2)$ . On the other hand, it is not hard to accept that whatever “expressing an antiderivative in elementary terms” means, erf isn’t it. In this article, my goal is to show that functions such as

$$f(x) = e^{-x^2}, \quad g(x) = \frac{e^x}{x}, \quad \text{and} \quad h(x) = \frac{\sin(x)}{x}$$

do not have elementary anti-derivatives. Naturally, the first task is to decide exactly what an elementary function is.

## 2. ELEMENTARY FUNCTIONS

The approach here is to use the language of field theory. We start with the field  $\mathcal{F}_0 = \mathbb{C}(z)$  of *complex* rational functions. Of course,  $\mathcal{F}_0$  is much too small—we know<sup>1</sup> of lots more functions than these, so naturally we'll add new functions by adjoining elements to  $\mathcal{F}_0$ . For example, we may want to add  $f(z) = \log(z)$ . Immediately we run into the question of domain. Here we will content ourselves with restricting attention to a suitable open subset  $\Omega$  of the plane. (Somewhat more generally, we could, for example, let  $\Omega$  be part of the usual Riemann surface for  $\log$ .) So, we assume that  $f$  is an analytic function on  $\Omega$  which satisfies  $e^{f(z)} = z$  for all  $z \in \Omega$  ( $f$  is called a *branch* of  $\log(z)$ ). Then we let  $\mathcal{K} = \mathcal{M}(\Omega)$  denote the *field* of meromorphic functions on  $\Omega$ . Then  $\mathbb{C}(z) \subset \mathcal{K}$  and we can define  $\mathcal{F}_1 = \mathcal{F}(f)$  to be the smallest subfield of  $\mathcal{K}$  containing the functions

$$g(z) = r_0(z) + r_1(z)f(z) + \cdots + r_n(z)(f(z))^n$$

for  $r_0, r_1, \dots, r_n \in \mathcal{F}_0$ . Naturally, if  $f_1, f_2, \dots, f_n \in \mathcal{M}(\Omega)$ , then we can iterate the process to obtain a subfield  $\mathbb{C}(z, f_1, \dots, f_n)$  of  $\mathcal{M}(\Omega)$ : that is,

$$\mathbb{C}(z) = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots \subseteq \mathcal{F}_n = \mathbb{C}(z, f_1, \dots, f_n),$$

where  $\mathcal{F}_{k+1} = \mathcal{F}_k(f_{k+1})$ .

We will be primarily interested in subfields  $\mathcal{E}$  of  $\mathcal{M}(\Omega)$  which are *differential* in the sense that  $\mathcal{E}' = \{f' : f \in \mathcal{E}\} \subseteq \mathcal{E}$ . Remarkably, many subfields are automatically differential.

**Lemma 1.** *Suppose that  $\mathcal{F}$  is a differential subfield of  $\mathcal{M}(\Omega)$  and that  $f \in \mathcal{M}(\Omega)$  is algebraic over  $\mathcal{F}$ . Then  $f' \in \mathcal{F}(f)$ ; that is,  $\mathcal{F}(f)$  is a differential subfield.*

*Proof.* If  $p(X) = r_0(z) + \cdots + r_m(z)X^m$  is the minimal polynomial of  $f$  over  $\mathcal{F}$ , then

$$(1) \quad f'(z) = \frac{r_0'(z) + r_1'(z)f(z) + \cdots + r_n'(z)(f(z))^m}{r_1(z) + 2r_2(z)f(z) + \cdots + nr_n(z)(f(z))^{m-1}}.$$

□

**Definition 2.** We say that a differential subfield of  $\mathcal{M}(\Omega)$  is an *elementary extension* of  $\mathbb{C}(z)$  on  $\Omega$  if  $\mathcal{E} = \mathbb{C}(z, f_1, \dots, f_n)$  for functions  $f_k \in \mathcal{M}(\Omega)$  such that either

- (1)  $f_{k+1}$  is algebraic over  $\mathcal{F}_k$ ,
- (2)  $\exp(f_{k+1})$  belongs to  $\mathcal{F}_k$ , or
- (3)  $f_{k+1} = \exp(g)$  for some  $g \in \mathcal{F}_k$ ,

for  $k = 0, 1, \dots, n-1$ . In case (2), we say that  $f_{k+1}$  is a logarithm of  $\mathcal{F}_k$ . Similarly, in case (3), we say that  $f_{k+1}$  is an exponential of  $\mathcal{F}_k$ .

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<sup>1</sup>One could speculate just how much we know about functions like  $f(z) = \log(z)$ , but we'll give that a pass here.

*Remark 3.* A subfield of  $\mathcal{M}(\Omega)$  of the form  $\mathbb{C}(z, f_1, \dots, f_n)$  with each  $f_k$  satisfying one of (1), (2), or (3) in Definition 2 is automatically differential and therefore elementary. To see this, it suffices to notice that  $\mathcal{F}_k(f_{k+1})$  is differential if  $\mathcal{F}_k$  is. However, this follows from Lemma 1 in case (1), and is clear in cases (2) and (3).

Finally, we can define an *elementary function*.

**Definition 4.** A function  $f$  in  $\mathcal{M}(\Omega)$  is elementary on  $\Omega$  if  $f$  belongs to some elementary extension of  $\mathbb{C}(z)$  on  $\Omega$ .

A few moments reflections shows that the definition of an elementary function is incredibly generous; it is difficult to conceive of a function which is not elementary.

*Example 5.* All the trigonometric functions and their inverses are elementary. For example:

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

and

$$\sin^{-1}(z) = -i \log(iz \pm \sqrt{1 - z^2})$$

are both elementary. The first is contained in  $\mathbb{C}(z, e^{iz})$ . The second is the logarithm of a function algebraic over  $\mathbb{C}(z)^2$ .

*Example 6.* Even complicated compositions are easily seen to be elementary:  $f(z) = \sin(\cos(z))$  is in  $\mathbb{C}(z, e^{iz}, e^{i \cos(z)})$ .

The above example is no accident, and is a general fact.

**Lemma 7.** *All (meromorphic)<sup>3</sup> composites of elementary functions are elementary.*

*Sketch of Proof.* Suppose that  $f$ ,  $g$ , and  $f \circ g$  are meromorphic on  $\Omega$ , and that  $f$  and  $g$  are elementary. We want to show that  $z \mapsto f(g(z))$  is elementary. Let  $f \in \mathbb{C}(z, f_1, \dots, f_n)$ . Since quotients of elementary functions are elementary, we can assume that  $f \in \mathbb{C}(z, f_1, \dots, f_{n-1})[f_n]$ . Thus it suffices to consider  $f(z) = r(z)(p(z))^k$  where  $r \in \mathcal{E} = \mathbb{C}(z, f_1, \dots, f_{n-1})$  and  $p$  is either algebraic over  $\mathcal{E}$  or a logarithm or exponential of  $\mathcal{E}$ . An induction argument allows us to assume that  $z \mapsto r(g(z))$  is elementary, and, since products of elementary functions are trivially elementary, we only have to see that  $z \mapsto p(g(z))$  is elementary.

First suppose that  $p$  is algebraic over  $\mathcal{E}$ . Then there are  $r_j, \dots, r_0 \in \mathcal{E}$  such that

$$r_j(z)(p(z))^j + \dots + r_0(z) = 0$$

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<sup>2</sup>It is *not* true that the inverse of an elementary function is elementary [1]. This does not appear to be easy to see and is certainly beyond the scope of this article

<sup>3</sup>We only look at compositions that are meromorphic. This is to avoid worrying about what happens near zero for atrocities like  $q(z) = 1/\sin(1/z)$ .

for all  $z \in \Omega$ . By induction, we can assume that each function  $z \mapsto r_i(g(z))$  belongs to some elementary extension  $\mathcal{E}'$ . Then  $z \mapsto p(g(z))$  is algebraic over  $\mathcal{E}'$ , and is elementary.

If  $p$  is a logarithm of  $\mathcal{E}$ , then by definition the function  $h(z) = \exp(p(z))$  is in  $\mathcal{E}$ , and induction once again implies that  $h \circ g$  is elementary and belongs to an elementary extension  $\mathcal{E}''$ . Then,  $z \mapsto p(g(z))$  is a logarithm of  $\mathcal{E}''$  and is elementary.

Finally, if  $p$  is an exponential of  $\mathcal{E}$ , say  $p(z) = \exp(h(z))$  for some  $h \in \mathcal{E}$ , we have  $h \circ g$  in an elementary extension  $\mathcal{E}''$  by induction, and  $z \mapsto p(g(z))$  is an exponential of  $\mathcal{E}''$ .  $\square$

*Example 8.* It follows that perfectly ridiculous functions like local solutions  $h$  to

$$(h(z))^5 \cos(\log(z)) + (h(z))^3 \tan(e^z + \sin^{-1}(z)) + 7 = 0$$

are elementary on their domains.

### 3. THE MAIN RESULTS

In spite of showing that virtually every conceivable function is elementary, we are going to prove the following theorem which asserts that many common anti-derivatives fail to be elementary—even if one is willing to sharply restrict the domain!

**Theorem 9.** *There is no open set  $\Omega$  on which the functions defined by  $\exp(-z^2)$ ,  $\exp(z)/z$ , and  $\sin(z)/z$  have elementary anti-derivatives. In fact, if  $f \in \mathbb{C}(z)$  and  $g \in \mathbb{C}(z)$  is non-constant, then  $z \mapsto f(z)\exp(g(z))$  has an elementary anti-derivative if and only if there is an  $a \in \mathbb{C}(z)$  such that  $f = a' + ag'$ .*

Our proof of this result is based on two rather difficult results. The first is a classical result of Liouville<sup>4</sup> suitably generalized by Ostrowski.

**Theorem 10 (Liouville).** *Let  $\mathcal{F} = \mathbb{C}(z)$  (or any other differential field of meromorphic functions containing  $\mathbb{C}(z)$ ) considered as a subfield of  $\mathcal{M}(\Omega)$  for some open subset  $\Omega$ . Then a function  $g \in \mathcal{F}$  has an elementary anti-derivative if and only if there are constants  $c_1, \dots, c_n \in \mathbb{C}$ , and functions  $u_1, \dots, u_n, v \in \mathcal{F}$  such that*

$$(2) \quad g = \sum_{i=1}^n c_i \frac{u_i'}{u_i} + v'.$$

The first step towards the proof of Theorem 10 is the following lemma which will also play a crucial rôle in the proof of the main theorem.

**Lemma 11.** *Suppose that  $g$  is a non-constant element of  $\mathbb{C}(z)$ . Then  $\theta = e^g$  is transcendental over  $\mathbb{C}(z)$ .*

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<sup>4</sup>Liouville developed his results in the period between 1833 and 1841. Actual references may be found in the bibliography of [1]

This lemma in turn depends on the following result which will also have applications elsewhere.

**Lemma 12.** *Suppose that  $\mathcal{F}$  is a differential subfield of  $\mathcal{M}(\Omega)$  containing  $\mathbb{C}(z)$ . Then if  $\sigma \in \text{Aut}(\mathcal{F}/\mathbb{C}(z))$ , then  $\sigma(\theta') = \sigma(\theta)'$  for all  $f\theta \in \mathcal{F}$ .*

*Proof.* Define  $D : \mathcal{F} \rightarrow \mathcal{F}$  by  $D(f) = f' - \sigma^{-1}(\sigma(f)')$ . It is not hard to see that  $D$  is a derivation on  $\mathcal{F}$  (i.e.,  $D(x+y) = D(x) + D(y)$  and  $D(xy) = D(x)y + xD(y)$ ), and that  $D(\mathbb{C}(z)) = \{0\}$ . Now if  $p$  is the minimal polynomial in  $\mathbb{C}(z)[X]$  with  $p(\theta) = 0$ , then

$$\begin{aligned} 0 = D(p(\theta)) &= p(\theta)' - \sigma^{-1}(p(\sigma(\theta))') \\ &= p'(\theta)\theta' - \sigma^{-1}(p'(\sigma(\theta))\sigma(\theta)') \\ &= p'(\theta)\theta' - p'(\theta)\sigma^{-1}(\sigma(\theta)') \\ &= p'(\theta)D(\theta). \end{aligned}$$

Since  $p'(\theta) \neq 0$ ,  $D(\theta) = 0$ ; this proves the claim<sup>5</sup>.  $\square$

*Proof of Lemma 11.* Suppose that  $\theta$  were not transcendental. Then  $\theta$  would be contained in a normal algebraic extension  $\mathcal{F}$  of  $\mathbb{C}(z)$  of degree  $n < \infty$ . Suppose that  $\sigma \in \text{Aut}(\mathcal{F}/\mathbb{C}(z))$ . Then Lemma 12 implies that  $\sigma(f)' = \sigma(f)'$  for all  $f \in \mathcal{M}(\Omega)$ , so that  $\sigma(\theta)' = \sigma(g'\theta) = g'\sigma(\theta)$ . Since the order of  $\text{Aut}(\mathcal{F}/\mathbb{C}(z))$  is  $n$ , and since  $g$  is not constant, we have

$$(3) \quad 0 \neq n \cdot g' = \sum_{\sigma} \frac{\sigma(\theta)'}{\sigma(\theta)} = \frac{(\prod_{\sigma} \sigma(\theta))'}{\prod_{\sigma} \sigma(\theta)}.$$

Put  $u = \prod_{\sigma} \sigma(\theta)$ . Note that if  $p$  is the minimal monic polynomial of  $\theta$  in  $\mathbb{C}(z)[X]$ , then  $p(x) = \prod_{\sigma} (X - \sigma(\theta))$ ; this shows that  $u \in \mathbb{C}(z)$ . It follows from (3) that  $u \neq 0$ .

Thus  $u'/u$  is in  $\mathbb{C}(z)$  and a derivative of  $n \cdot g$  in  $\mathbb{C}(z)$ . The last statement implies that all the poles of  $u'/u$  have order 2. On the other hand the fact that  $u$  is non-constant implies that  $u'/u$  has poles, and any such poles must be of order 1. This contradiction completes the proof.  $\square$

Liouville's original work depended on a rather subtle analysis of the structure of elementary functions. This is exploited in [1], but, at least I found, the language and style involved there to be rather difficult to absorb. The proofs given here will be based on the following result of Rosenlicht [2].

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<sup>5</sup>This is essentially a proof that differentiation is the unique derivation extending differentiation on any algebraic extension of  $\mathbb{C}(z)$ .

**Proposition 13 (Rosenlicht).** *Assume that  $\mathcal{F}$  is a differential subfield of  $\mathcal{M}(\Omega)$ . Suppose that  $\theta \in \mathcal{M}(\Omega)$  is transcendental over  $\mathcal{F}$ , and that either  $\theta' \in \mathcal{F}$  or  $\theta'/\theta \in \mathcal{F}$ . Finally suppose that  $c_1, \dots, c_n \in \mathcal{F}$  are linearly independent over  $\mathbb{Q}$ , that  $v, u_1, \dots, u_n \in \mathcal{F}(\theta)$ , and that*

$$\sum_{i=1}^n c_i \frac{u_i'}{u_i} + v' \in \mathcal{F}[\theta].$$

*Then  $v$  is actually in  $\mathcal{F}[\theta]$ . Furthermore, if  $\theta' \in \mathcal{F}$ , then each  $u_i$  must be in  $\mathcal{F}$ . On the other hand, if  $\theta'/\theta \in \mathcal{F}$ , then for each  $i = 1, \dots, n$ , there is an integer  $\nu_i$  such that  $u_i/\theta^{\nu_i} \in \mathcal{F}$ .*

We will postpone the proof of Proposition 13 until the very last. However our main result is now a straightforward consequence of Theorem 10 and Proposition 13.

*Proof of Theorem 9.* Suppose that  $\alpha(z) = f(z) \exp(g(z))$  with  $f, g \in \mathbb{C}(z)$ ,  $f \neq 0$ , and  $g \notin \mathbb{C}$ . If  $\alpha$  has an elementary anti-derivative, then we can apply Theorem 10 with  $\mathcal{F} = \mathbb{C}(z, e^g)$ . Thus

$$(4) \quad fe^g = \sum_{i=1}^n c_i \frac{u_i'}{u_i} + v'$$

with  $c_1, \dots, c_n \in \mathbb{C}$  and  $v, u_1, \dots, u_n \in \mathbb{C}(z, e^g)$ .

**Lemma 14.** *We may assume that the constants  $c_1, \dots, c_n$  appearing in (4) are linearly independent over  $\mathbb{Q}$ .*

*Proof.* Otherwise we may as well assume that  $c_n = (m_1 c_1 + \dots + m_{n-1} c_{n-1})/m$  with  $m_1, \dots, m_n, m$  integers and  $m \neq 0$ . For  $1 \leq i \leq n-1$ , let  $w_i = u_i^m$ ,  $d_i = c_i/m$ , and  $W - n = u_n$ . Then if we put  $d_n = c_n$ , we have  $d_n = m_1 d_1 + \dots + m_{n-1} d_{n-1}$ , and  $w_i'/w_i = \frac{m u_i^{m-1} u_i'}{u_i^m} = m \frac{u_i'}{u_i}$ , for  $1 \leq i \leq n-1$ . With these notations,

$$\begin{aligned} \alpha &= \sum_{i=1}^n d_i \frac{w_i'}{w_i} + v' = \sum_{i=1}^{n-1} d_i \left( \frac{w_i'}{w_i} + \frac{m_i w_n'}{w_n} \right) + v' \\ &= \sum_{i=1}^{n-1} d_i \frac{(w_i w_n^{m_i})'}{w_i w_n^{m_i}} + v'. \end{aligned}$$

□

It follows from Lemma 11 that  $e^g$  is transcendental over  $\mathbb{C}(z)$ . Thus we may apply Proposition 13 with  $\mathcal{F} = \mathbb{C}(z)$  and  $\theta = e^g$  to conclude that  $v \in \mathbb{C}(z)[e^g]$  and

$u_i = h_i e^{\nu_i g}$  for some functions  $h_i \in \mathbb{C}(z)$  and integers  $\nu_i$ . In particular,  $u'_i/u_i \in \mathbb{C}(z)$  and

$$(5) \quad f e^g + v' \in \mathbb{C}(z).$$

On the other hand,  $v = \sum_{j=0}^m b_j e^{jg}$  with  $b_i \in \mathbb{C}(z)$  and  $b_m \neq 0$ . Thus

$$(6) \quad v' = \sum_{j=0}^m (b'_j + j b_j g') e^{jg}.$$

Notice that if both  $j$  and  $b_j$  are nonzero, then having  $b'_j + j b_j g' = 0$  would imply that  $g'$  had poles of order 1—a contradiction<sup>6</sup>. Thus for each  $j \geq 1$ , we have  $(b'_j + j b_j g') \neq 0$  whenever  $b_j \neq 0$ . However as  $e^g$  is transcendental over  $\mathbb{C}(z)$ , we certainly have  $\{1, e^g, e^{2g}, \dots, e^{mg}\}$  linearly independent over  $\mathbb{C}(z)$ . Therefore (5) implies that  $m = 1$ , and  $f$  must be the coefficient of  $e^g$  in (6). That is,  $f = a' + a g'$  with  $a \in \mathbb{C}(z)$ . This proves the last statement in the Theorem as  $z \mapsto a(z) \exp(g(z))$  would be an anti-derivative of  $\alpha$  in the event  $f = a' + a g'$ .

If  $\alpha(z) = \exp(-z^2)$ , then we could apply the above with  $f(z) = 1$  and  $g(z) = -z^2$ . Therefore  $\alpha$  has an elementary anti-derivative if and only if there is a rational function  $a$  such that

$$(7) \quad 1 = a'(z) - 2z a(z).$$

However,  $a$  has a partial fraction decomposition of the form

$$(8) \quad a(z) = p(z) + \sum_{k=1}^m \sum_{r=1}^{n_r} \frac{a_r^m}{(z - z_k)^r},$$

where  $p$  is a polynomial and  $a$  has poles  $z_1, \dots, z_m$  of multiplicities  $n_1, \dots, n_m$  (with  $n_i \geq 1$  for all  $i$ ). Clearly,  $p = 0$  as otherwise the right-hand side of (7) has degree equal to  $\deg(p) + 1 > 0$ . On the other hand, if  $a$  has a pole of order  $s \geq 1$  at  $z_0$ , then the right-hand side of (7) has a pole of order  $s + 1$ . This is a contradiction, and we conclude that  $\exp(-z^2)$  does not have an elementary anti-derivative.

If  $\alpha(z) = (1/z) \exp(z)$ , so that  $f(z) = 1/z$  and  $g(z) = z$ . This requires that we solve

$$(9) \quad \frac{1}{z} = a'(z) + a(z).$$

However, the right-hand side of (9) has no poles of order one. Thus  $\alpha$  can't have an elementary integral. In fact, neither does  $z \mapsto (1/z) \exp(w_0 z)$  for any non-zero constant  $w_0 \in \mathbb{C}$ .

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<sup>6</sup>Because  $g' = -\frac{b'_j}{j b_j}$  must be nonzero, since  $g$  is nonconstant, and hence has poles which must be of order one.

Finally we are left with the question of whether  $q(z) = \sin(z)/z$  has an elementary integral. Note that if we define  $\theta(z) = \exp(iz)$ , then  $q(z) = (1/2iz)(\theta(z) - \theta(z)^{-1})$ . Furthermore, we have just concluded that neither  $\theta/z$  nor  $1/(z\theta)$  have elementary anti-derivatives. Unfortunately, it does not immediately follow from this that  $q$  can not have an elementary anti-derivative!

We need to introduce some notation and conventions for functions in  $v \in \mathbb{C}(z, \theta)$ . These functions are rational in  $\mathbb{C}(z)[\theta]$ ; that is  $v$  is a quotient of polynomials in  $\theta(z)$  with coefficients in  $\mathbb{C}(z)$ . We use the notation  $v_i(z, \theta(z))$  to denote a function of the form  $p(\theta(z))/q(\theta(z))$  for  $p, q \in \mathbb{C}(z)[X]$ . In particular, we can view  $v$  as a function of two independent variables where  $v(z, w) = p(w)/q(w)$  (keep in mind that the coefficients of  $p$  and  $q$  are functions of  $z$ ). Thus we can write  $D_1v$  for the partial derivative of  $v$  with respect to its first variable, and  $D_2v$  for the partial with respect to the second<sup>7</sup>.

Now back to  $q$ . If  $q$  did have an elementary integral  $u$ , then Theorem 10 would allow us to assert that there are functions  $v_0, v_1, \dots, v_r$ , which are rational in  $\mathbb{C}(z)[\theta]$ , such that

$$u(z) = v_0(z, \theta(z)) + \sum_{i=1}^r c_i \log(v_i(z, \theta(z))),$$

and the ordinary chain rule implies that

$$(10) \quad \frac{1}{2iz} \left( \theta(z) - \frac{1}{\theta(z)} \right) = D_1v_0(z, \theta(z)) + D_2v_0(z, \theta(z))i\theta(z) \\ + \sum_{i=1}^r c_i \frac{D_1v_i(z, \theta(z)) + D_2v_i(z, \theta(z))i\theta(z)}{v_i(z, \theta(z))}$$

Since  $\theta$  is transcendental over  $\mathbb{C}(z)$  (Lemma 11), (10) must hold with  $\theta(z)$  replaced by any value  $\zeta$  in the domain of both sides. In particular, we can replace  $\theta(z)$  by  $\mu\theta(z)$  for any  $\mu$  sufficiently close to 1. It follows that

$$\frac{1}{2iz} \left( \mu\theta(z) - \frac{1}{\mu\theta(z)} \right)$$

is the derivative of  $u_\mu(z) = v_0(z, \mu\theta(z)) + \sum_i c_i \log(v_i(z, \mu\theta(z)))$ . This would imply that

$$\frac{i\mu}{\mu^2 - 1}(u + u_\mu)$$

is an elementary anti-derivative of  $\theta(z)/z$ , and this is impossible. This contradiction completes the proof.  $\square$

<sup>7</sup>For example, suppose that  $q = 1$  and  $p(X) = r_0(z) + r_1(z)X + \dots + r_k(z)X^k$ . Then  $D_1v(z, w) = r'_0(z) + r'_1(z)w + \dots + r'_k(z)w^k$ , while  $D_2v(z, w) = r_1(z) + \dots + kr_k(z)w^{k-1}$



The last bit about  $\sin(z)/z$  is actually a very special case of a generalization of Theorem 10 given in [1, III §13] which says that, assuming  $g_i$  and  $y_i$  are algebraic over  $\mathbb{C}(z)$  and no  $g_i$  is a constant multiple of another  $g_j$ , then  $w = e^{g_1}y_1 + \cdots + e^{g_n}y_n$  has an elementary anti-derivative only if each  $e^{g_i}y_i$  does. This is beyond our current scope.

Now we have to pay for our fun and prove the two hard results, Theorem 10 and Proposition 13, on which all the above depends.

*Proof of Theorem 10.* By assumption (and Remark 3) there is a tower of *differential* fields

$$\mathcal{F} = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_N,$$

such that  $g \in \mathcal{F}'_N$ , and for each  $i = 1, 2, \dots, N$ ,  $\mathcal{F}_i = \mathcal{F}_{i-1}(f_i)$  where, if  $f_i$  is not algebraic over  $\mathcal{F}_{i-1}$ , then  $f_i$  is either a logarithm or exponential of  $\mathcal{F}_{i-1}$ .

Notice that if  $N = 0$ , then the result is trivially true. So we can proceed by induction on  $N$ . So we can assume that  $N > 0$  and apply the induction hypothesis to the tower  $\mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_N$ . Consequently, we can assume that

$$(11) \quad g = \sum_{i=1}^r c_i \frac{u'_i}{u_i} + v',$$

where  $c_1, \dots, c_r \in \mathbb{C}$  as desired, but  $v, u_1, \dots, u_r$  are only known to be in  $\mathcal{F}_1$ . Our task is to show that we can modify  $r, c_1, \dots, c_r, u_1, \dots, u_r$  such that  $u_1, \dots, u_r \in \mathcal{F}$ .

For notational convenience, we write  $\theta = f_1$  so that  $\mathcal{F}_1 = \mathcal{F}(\theta)$ . Also, the argument of Lemma 14 implies that we may assume that the  $c_i$  are linearly independent over  $\mathbb{Q}$ . (This will be needed when we try to apply Proposition 13.)

We first concentrate on the case where  $\theta$  is transcendental over  $\mathcal{F}$ . If  $\theta$  is logarithm of  $\mathcal{F}$ , then  $\theta' = a'/a$  for some  $a \in \mathcal{F}$ . We apply Rosenlicht's Proposition 13 which implies that each  $u_i$  belongs to  $\mathcal{F}$  and that  $v$  belongs to  $\mathcal{F}[\theta]$ . Then (11) also forces  $v'$  to belong to  $\mathcal{F}$ . We will have obtained the result in the present case if we can show all this forces  $v$  to be of the form  $c\theta + d$  with  $c \in \mathbb{C}$  and  $d \in \mathcal{F}$ . But

$$(12) \quad v = \sum_{j=0}^m b_j \theta^j,$$

with  $b_j \in \mathcal{F}$ . We can assume that  $m > 0$ , and that  $b_m \neq 0$  (otherwise we're done). Anyway,

$$v' = b'_m \theta^m + \left( m b_m \frac{a'}{a} + b'_{m-1} \right) \theta^{m-1} + (\text{an element of } \mathcal{F}[\theta] \text{ of degree } < m - 1).$$

Since  $v' \in \mathcal{F}$ , we get  $b'_m = 0$ , so  $b_m$  is a constant function. Similarly, if  $m > 1$ , then we also have  $0 = (m b_m a'/a + b'_{m-1})$ , which equals  $(m b_m \theta + b_{m-1})'$ . Thus  $(m b_m \theta + b_{m-1})'$

is a constant and belongs to  $\mathcal{F}$ . This contradicts the transcendency of  $\theta$  over  $\mathcal{F}$ , and completes the proof in the logarithmic case.

Now suppose that  $\theta$  is an exponential of  $\mathcal{F}$ . Then  $\theta'/\theta = a'$  for some  $a \in \mathcal{F}$ . Again we can apply Proposition 13 to conclude that each  $u_i$  is of the form  $a_i\theta^{\nu_i}$  for  $a \in \mathcal{F}$  and  $\nu_i$  an integer, while  $v \in \mathcal{F}[\theta]$ . Now calculate that

$$\frac{u_i'}{u_i} = \frac{a_i'}{a_i} + \nu_i \frac{\theta'}{\theta} = \frac{a_i'}{a_i} + \nu_i a',$$

and plugging into (11) we get

$$g = \sum_{i=1}^r c_i \frac{a_i'}{a_i} + (v + (\nu_1 + \cdots + \nu_r)a)'.$$

Thus we may assume that each  $u_i$  actually belongs to  $\mathcal{F}$ . This again forces  $v'$  to belong to  $\mathcal{F}$ , and we only need to see that  $v \in \mathcal{F}$  too. But  $v$  has the form (12) with each  $b_j$  in  $\mathcal{F}$  and  $b_m \neq 0$ . In this case,

$$v' = \sum_{j=0}^m (b_j' + j b_j a') \theta^j.$$

If  $m \neq 0$ , we have  $b_m' - m b_m a' = 0$ . This implies that

$$\begin{aligned} (b_m \theta^m)' &= b_m' \theta^m + b_m m \theta^{m-1} \theta' = \theta^m \left( b_m' + m b_m \frac{\theta'}{\theta} \right) \\ &= \theta^m (b_m' + m b_m a') = 0. \end{aligned}$$

This forces  $b_m \theta^m$  to belong to  $\mathcal{F}$ , which is nonsense. This disposes of the cases where  $\theta$  is transcendental over  $\mathcal{F}$ .

Finally, we are left only with the case that  $\theta$  is algebraic over  $\mathcal{F}$ . Let  $\mathcal{K}$  be the smallest normal extension of  $\mathcal{F}$  containing  $\mathcal{F}_1$ . For any  $\sigma$  in  $\text{Aut}(\mathcal{K}/\mathcal{F})$  we have

$$g = \sum_{i=1}^r c_i \frac{\sigma(u_i)'}{\sigma(u_i)} + \sigma(v') + \sum_{i=1}^r c_i \frac{\sigma(u_i)'}{\sigma(u_i)} + (\sigma(v)').$$

(We saw in the proof of Lemma 11 that  $\sigma(f)' = \sigma(f')$  for all  $f \in \mathcal{K}$ .) Therefore

$$[\mathcal{K} : \mathcal{F}]g = \sum_{i=1}^r \sum_{\sigma} \frac{\sigma(u_i)'}{\sigma(u_i)} + \sum_{\sigma} \sigma(v)' = \sum_{i=1}^r c_i \frac{(\prod_{\sigma} \sigma(u_i))'}{\prod_{\sigma} \sigma(u_i)} + \left( \sum_{\sigma} \sigma(v) \right)'.$$

Since both  $\prod_{\sigma} \sigma(u_i)$  and  $\prod_{\sigma} \sigma(v)$  are invariant under  $\text{Aut}(\mathcal{K}/\mathcal{F})$ , they are in  $\mathcal{F}$ . Thus we can divide both sides by  $[\mathcal{K} : \mathcal{F}]$  to get the desired result.  $\square$

Before I tackle Rosenlicht's fundamental result, I'll need a little lemma.

**Lemma 15.** *Suppose that  $p_1, \dots, p_N$  are distinct irreducibles in  $\mathcal{K}[\theta]$ <sup>8</sup>. Also suppose that  $\{a_{mk}\}_{m=1, k=1}^{N, n_m}$  are polynomials in  $\mathcal{K}[\theta]$  such that  $p_m$  and  $a_{mn_m}$  are relatively prime for all  $m = 1, \dots, N$ . Then*

$$(13) \quad \sum_{m=1}^N \sum_{k=1}^{n_m} \frac{a_{mk}}{p_m^k}$$

is not in  $\mathcal{K}[\theta]$ .

*Proof.* Let  $M = \max_m n_m$ . If (13) were in  $\mathcal{K}[\theta]$ , then we could multiply by  $(p_1 p_2 \dots p_N)^M$  to conclude that is a subset of  $\{p_m\}, q_1, \dots, q_{N'}$ , such that

$$(14) \quad \sum_{m=1}^{N'} \frac{a_m}{q_m}$$

is in  $\mathcal{K}[\theta]$  with  $0 < N' \leq N$  and each  $a_m$  relatively prime to each  $q_m$ . But (14) equals

$$(15) \quad \frac{a_1 \prod_{m=2}^{N'} q_m + q_1 b}{\prod_{m=1}^{N'} q_m}$$

for some  $b \in \mathcal{K}[\theta]$ . Thus if (14) is in  $\mathcal{K}[\theta]$ , then  $q_1$  divides the numerator of (15), and therefore  $q_1$  divides  $a_1 \prod_{m=2}^M q_m$ . This is a contradiction and completes the proof.  $\square$

*Proof of Proposition 13.* We let  $\mathcal{K}$  be a finite normal algebraic extension of  $\mathcal{F}$  in which the functions  $v, u_1, \dots, u_n$  split into linear factors. That is, we assume that

$$u_i = g_i \prod_{j=1}^N (\theta - z_j)^{\mu_{ij}}, \quad i = 1, 2, \dots, n$$

$$v = \sum_{j=1}^N \sum_{\nu=-1}^{-M} h_{\nu j} (\theta - z_j)^\nu + \text{an element of } \mathcal{K}[\theta],$$

where  $h_{\nu j}$  and  $z_j$  are elements of  $\mathcal{K}$ , each  $g_i$  is a nonzero element of  $\mathcal{K}$ , and  $\mu_{ij}$  is an integer. Note that  $\mu_{ij}$  and  $h_{\nu, j}$  will often be zero. By an argument only slightly more complicated than that in Remark 3, we see that  $\mathcal{K}$  is differential<sup>9</sup>, and therefore so is  $\mathcal{K}(\theta)$ . Moreover,  $\mathcal{K}[\theta]$  is closed under differentiation as well.

With these notations, our basic assumption is that

$$(16) \quad \sum_{i=1}^n c_i \frac{g_i'}{g_i} + \sum_{i,j} c_i \mu_{ij} \frac{\theta' - z_j'}{\theta - z_j} + \sum_{j,\nu} (h_{\nu j} (\theta - z_j)^\nu)' \in \mathcal{K}[\theta].$$

<sup>8</sup>We could replace  $\mathcal{K}[\theta]$  by any UFD, and  $\mathcal{K}(\theta)$  by its fraction field.

<sup>9</sup>For each  $f \in \mathcal{K}$ , consider the minimal polynomial  $p \in \mathcal{F}[X]$  of  $f$ . If  $p(X) = r_0(z) + \dots + r_m(z)X^m$  for functions  $r_i \in \mathcal{F}$ , then  $f'(z)$  is given by the formula (1). Thus  $f' \in \mathcal{K}$ .

I intend to show that, unless  $z_j = 0$ , all the  $\mu_{ij}$  and  $h_{\nu j}$  are zero. This will be shown to suffice.

We have either  $\theta' = a$  or  $\theta' = a\theta$  for some  $a \in \mathcal{F}$ . We will consider that case  $\theta' = a$  first. Then

$$(17) \quad \frac{\theta' - z_j}{\theta - z_j} = \frac{a - z_j}{\theta - z_j}.$$

Next I claim that (17) is in lowest terms. (Recall that  $p/q$  is in lowest terms if  $p$  and  $q$  are relatively prime in  $\mathcal{K}[\theta]$ .) Since the denominator is irreducible, this amounts to showing that  $a - z_j \neq 0$ . But, if  $a = z'_j$ , then  $a = \sigma(z'_j) = \sigma(z_j)'$  for all  $\sigma \in \text{Aut}(\mathcal{K}/\mathcal{F})$  (see the proof of Lemma 11). Therefore  $[\mathcal{K} : \mathcal{F}]a = (\sum_{\sigma} \sigma(z_j))'$ , and  $a = b'$  for some  $b \in \mathcal{F}$ . This implies that  $(\theta - b)' = 0$ , so that  $\theta - b$  is a constant and  $\theta \in \mathcal{F}$ . This is nonsense, and establishes the claim.

In the case  $\theta' = a\theta$ , we have

$$\frac{\theta' - z_j}{\theta - z_j} = \frac{a\theta - z_j}{\theta - z_j},$$

and again I claim that the quotient is in lowest terms, provided that  $z_j \neq 0$ . Here that means that  $z'_j \neq az_j$ . If not, then we would have  $z'_j/z_j = a$ , and once again  $\sigma(z_j)'/\sigma(z_j) = a$  for all  $\sigma \in \text{Aut}(\mathcal{K}/\mathcal{F})$ . Then

$$[\mathcal{K} : \mathcal{F}]a = \sum_{\sigma} \frac{\sigma(z_j)'}{\sigma(z_j)} = \frac{(\prod_{\sigma} \sigma(z_j))'}{\prod_{\sigma} \sigma(z_j)}.$$

Letting  $D = [\mathcal{K} : \mathcal{F}]$ , we have  $Da = b'/b$  for some non-zero  $b$  in  $\mathcal{F}$ . Thus

$$\frac{(\theta^D)'}{\theta^D} = D \frac{\theta'}{\theta} = \frac{b'}{b},$$

which implies that  $(\theta^D/b)' = 0$ . This, in turn, implies that  $\theta^D/b \in \mathcal{F}$  which is a contradiction. Thus we have shown that in all cases but one, the fraction  $(\theta' - z_j)/(\theta - z_j)$  is in lowest terms. The exceptional case occurs when  $\theta'/\theta \in \mathcal{F}$  and  $z_j = 0$ . Then  $(\theta' - z_j)/(\theta - z_j) = \theta'/\theta \in \mathcal{F}$ .

Now consider

$$(18) \quad (h_{\nu j}(\theta - z_j)^{\nu})' = h'_{\nu j}(\theta - z_j)^{\nu} + \nu h_{\nu j}(\theta - z_j)^{\nu-1}(\theta' - z'_j).$$

As above, except in the exceptional case or when  $h_{\nu j} = 0$ , (18) is of the form  $\mathcal{K}[\theta]$  plus a fraction which, in lowest terms, had a linear numerator and a denominator of degree  $-\nu + 1 > 1$ . In the exceptional case where  $z_j = 0$  and  $\theta'/\theta = a \in \mathcal{F}$ , we have

$$(h_{\nu j}(\theta - z_j)^{\nu})' = \theta^{\nu}(h'_{\nu j} + \nu h_{\nu j}a).$$

I claim that if  $h_{\nu j} \neq 0$ , then also  $h'_{\nu j} + \nu h_{\nu j} a \neq 0$ . If the claim were false, then we have  $h'_{\nu j}/h_{\nu j} = -\nu a$ , and  $(\sigma(h_{\nu j}))'/\sigma(h_{\nu j}) = -\nu a$  for all  $\sigma \in \text{Aut}(\mathcal{K}/\mathcal{F})$ . Once again

$$-[\mathcal{K} : \mathcal{F}] \nu a = \sum_{\sigma} \frac{\sigma(h_{\nu j})'}{\sigma(h_{\nu j})} = \frac{(\prod_{\sigma} \sigma(h_{\nu j}))'}{\prod_{\sigma} \sigma(h_{\nu j})},$$

so that  $-N \nu a = -b/b$  with  $N = [\mathcal{K} : \mathcal{F}]$  and  $0 \neq b \in \mathcal{F}$ . But then  $(\theta^{-N\nu})'/\theta^{-N\nu} = -N \nu a = b'/b$ , and  $(\theta^{-N\nu}/b)' = 0$ . This implies that  $\theta^{-N\nu}/b \in \mathcal{F}$ , which is a contradiction. So in the exceptional case, (18) has a fractional part which, in lowest terms, has a denominator of degree  $-\nu > 0$ . It follows from Lemma 15 that if any  $h_{\nu j} \neq 0$ , then (16) can not reduce to a polynomial in  $\mathcal{K}[\theta]$ . In particular,  $v \in \mathcal{K}[\theta] \cup \mathcal{F}(\theta) = \mathcal{F}[\theta]$ .

Thus we are left with

$$\sum_{\substack{i,j \\ z_j \neq 0}} c_i \mu_{ij} \frac{\theta' - z'_j}{\theta - z_j} \in \mathcal{K}[\theta].$$

Using Lemma 15 again, it follows that  $\sum_i c_i \mu_{ij} = 0$  whenever  $z_j \neq 0$ . Since the  $c_i$  are independent over  $\mathbb{Q}$  by assumption, we have  $\mu_{ij} = 0$  unless  $z_j = 0$  (and  $\theta'/\theta \in \mathcal{F}$ ). In any case, there are integers  $\nu_1, \dots, \nu_n$  (all zero in the event  $\theta' \in \mathcal{F}$  and equal to  $\mu_{ij}$  when  $z_j = 0$  and  $\theta'/\theta \in \mathcal{F}$ ) such that  $u_i/\theta^{\nu_i} \in \mathcal{K}$ . But then  $u_i/\theta^{\nu_i} \in \mathcal{K} \cap \mathcal{F}(\theta) = \mathcal{F}$ . This completes the proof of Rosenlicht's result, and completes our investigation.  $\square$

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