DARTMOUTH COLLEGE
Hanover, New Hampshire

G-EDGE COLORED PLANAR ROOK ALGEBRA

A thesis submitted in partial
requirement for a

Bachelor of Arts
with high honors

in

Mathematics

by

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Contents

Abstract

In this thesis, we study the Type B Planar Rook Monoid PR_k^B and give a set of generators and relations for it. We then study the regular representation of the G-edge colored version of the Planar Rook Algebra $PR_k(n;G)$ for a group G and completely decompose the regular representation in the case that G is a finite abelian group and show that in this case $PR_k(n;G)$ is a semisimple algebra. We then determine the branching rules and define an indexing set for the irreducible representations of $PR_k(n;G)$ using combinatorial objects. Then we present an example of a small nonabelian group G for which $PR_1(n;G)$ is not a semisimple algebra.

Acknowledgement

I would like to thank my advisor, Professor Rosa Orellana, for all of her support throughout the process of writing this thesis. She has pushed me from day one to write better and to present my work in the clearest way possible. She has also battled time constraints and sickness to help me through the last few weeks of writing this thesis and putting together the presentation. I would also like to thank Zajj Daugherty and President Phil Hanlon for several helpful conversations on the centralizer algebra of the Rook Algebra, which unfortunately is not part of this thesis. Credit goes to Hanh Nguyen for the title page, who is a wizard with LaTeX. I would like to thank Hanh Nguyen, Jacob Richey and Sam Aney for a wonderful four years full of math, friendship, and a countable number of Putnam problems.

Chapter 1

Introduction

Let S_n be the group of permutations of $[n]$ and V be the permutation representation of S_n. Martin [8] and Jones [7] independently studied the centralizer algebra $\mathrm{End}_{S_n}(V^{\otimes k})$, the algebra of endomorphisms of $V^{\otimes k}$ which commute with the diagonal action of S_n on $V^{\otimes k}$. This algebra, called the Partition Algebra, has been studied extensively by various mathematicians including Halverson and Ram in [5]. This thesis studies the Representation Theory of group-edge colorings of subalgebras of the Partition Algebra.

We can view the symmetric group S_k in terms of diagrams and then use this to act on $V^{\otimes k}$. By Classical Schur-Weyl duality, the centralizer of S_k under this action is then the general linear group $GL_n(\mathbb{C})$. In Chapter 3, we then take the set of subdiagrams of these permutation diagrams where we remove edges and define a monoid structure on this set, which is called the Rook Monoid R_k and has been studied by [4]. If we then consider only those diagrams whose edges do not cross, we obtain a submonoid of R_k, the Planar Rook Monoid, which we denote PR_k. We can then consider the span of the Planar Rook diagrams and define the subalgebra $PR_k(n)$, which is a subalgebra of the Partition Algebra. The special case when $n = 1$ has been studied by Herbig in [6].

In [1], Bloss studies the centralizer of the wreath product $G \wr S_n$ of a group G with S_n, $\mathrm{End}_{G \wr S_n}(V^{\otimes k})$, and characterizes it as a diagram algebra consisting of partition diagrams whose edges are oriented and labeled with elements of G. This algebra is called the G-edge colored Partition Algebra $P_k(n;G)$. In Chapter 4, we study the regular representation of the subalgebra of $P_k(n;G)$ consisting of Planar Rook diagrams, which we denote by $PR_k(n;G)$, for finite abelian groups G. Note that we no longer have to consider the edges oriented when we restrict to these diagrams since we can assume that all edges are oriented upward. We find a complete decomposition of the regular representation of $PR_k(n;G)$ in order to show that the algebra is semisimple and to find all of its finite-dimensional irreducible representations. We then determine which irreducible subrepresentations of the regular representation are distinct and determine how each of them decomposes into irreducible representation after restricting the action to $PR_{k-1}(n;G)$ and draw the Bratteli Diagram for the case $G = \mathbb{Z}_2$.

Mousley, Schley and Shoemaker study the Planar Rook Algebra colored with r colors in [9]. Note that this algebra is distinct from the $\mathbb{Z}_r$-edge colored Planar Rook Algebra. In the Planar Rook Algebra colored with r colors, multiplication of colored diagrams is defined so that when two edges colored with different colors meet they cancel each other out. In the $\mathbb{Z}_r$-edge colored Planar Rook Algebra, these edges form an edge labeled with the product of the group elements corresponding to the two edges.

Chapter 2

Preliminaries

2.1 Representation Theory

In this section, we will give the basic preliminary definitions and theorems utilized in this thesis. For more detailed descriptions and examples, see [2]. Let us start with the definition of an associative algebra, which is the primary mathematical object this thesis is concerned with.

Definition 2.1.1. An *associative algebra* over a field F is a vector space A over F equipped with a multiplication operation $A \times A \to A$, which we write as juxtaposition $(a, b) \mapsto ab$, which is associative and bilinear.

For the purposes of this thesis, all algebras are assumed to be associative and containing a multiplicative identity.

An example of an algebra is the *group algebra* over a field F of a group G, denoted by $F[G]$ or $\mathbb{C}G$ in the case of $F = \mathbb{C}$, is the algebra generated by the F-span of the set $\{x_g \,|\, g \in G\}$ where multiplication is defined as $x_g x_{g'} = x_{gg'}$ and extended linearly. Often, we write g in place of x_g.

Given two algebras A and B over the field F, an *algebra homomorphism* from A to B is a linear map $\phi : A \to B$ preserving the multiplication operation and sending the multiplicative identity of A to the multiplicative identity of B.

A *representation* of an algebra A over a field F (also called a left A-module) is a F-vector space V together with an algebra homomorphism $\rho : A \to \operatorname{End} V$, where $\operatorname{End} V$ is the algebra of linear maps from V to itself. For $a \in A$ and $v \in V$, $\rho(a)(v)$ is usually denoted by av. This thesis will be concerned with classifying all representations of an algebra. We will also look at *subrepresentations* of a representation V of A, which are subspaces $U \subseteq V$ which are invariant under all linear maps $\rho(a)$ for all $a \in A$. Given a subrepresentation U of V, we may define a new algebra homomorphism $\rho' : A \to \operatorname{End} U$ which takes each $a \in A$ to the restriction of $\rho(a)$ to U.

A representation V is *irreducible* if its only subrepresentations are 0 and V itself. This thesis will mainly be concerned with classifying all irreducible representations of our algebra. To do this, we will look at the *regular representation* of the algebra A, which is the representation with $V = A$ and $\rho(a)(b) = ab$ for a and b in A, where ab is the product of a and b in the algebra A.

The *radical* of a finite dimensional algebra A, denoted $\mathrm{Rad}(A)$, is the set of all elements of A which act by 0 in all irreducible representations of A. We then call the algebra *semisimple* if $\mathrm{Rad}(A) = 0$.

A representation is completely reducible if it can be expressed as the direct sum of irreducible representations of A.

By the following proposition, finding a complete decomposition of our algebra will tell us a lot about A and its irreducible representations.

Proposition 2.1.2. *[3, Proposition 2.16] For a finite dimensional algebra A over field F, the following are equivalent:*

1. *A is semisimple.*
2. *$\sum_i (\mathrm{dim} V_i)^2 = \mathrm{dim} A$, where the V_i's are the irreducible representations of A.*
3. *$A \cong \oplus_i \mathrm{Mat}_{d_i}(F)$ for some d_i, a direct sum of matrix algebras with entries in F.*
4. *Any finite dimensional representation of A is completely reducible.*
5. *The regular representation of A is a completely reducible representation.*

The *tensor product* $V \otimes W$ of two vector spaces V and W over field F is the quotient of the space whose basis is given by the formal symbols $v \otimes w$ for $v \in V$ and $w \in W$ by the subspace spanned by the elements

i. $(cv) \otimes w - c(v \otimes w)$

ii. $v \otimes (cw) - c(v \otimes w)$

iii. $(v + v') \otimes w - v \otimes w - v' \otimes w$

iv. $v \otimes (w + w') - v \otimes w - v \otimes w'$

for all $c \in F$ and $v, v' \in V$ and $w, w' \in W$

We can then define the k^{th} tensor power of V, $V^{\otimes k} = V \otimes \cdots \otimes V$ (k copies of V). If V and W are representations of an algebra A with actions ρ_V and ρ_W, then the representation $V \otimes W$ of A is defined by the action

$$\rho(a)(v \otimes w) = \rho_V(a)(v) \otimes \rho_W(a)(w).$$

for each $a \in A$.

The *wreath product* of a finite group G with the symmetric group S_n, denoted by $G \wr S_n$, is the group with underlying set given by the Cartesian product $G^n \times S_n$, where G^n is the direct product of n copies of G, and multiplication given by

$$(g, \sigma)(h, \tau) = (g(\sigma(h)), \sigma\tau)$$

where $\sigma(h)$ is the element of G^n whose entries are the entries of h permuted by σ.

Chapter 3

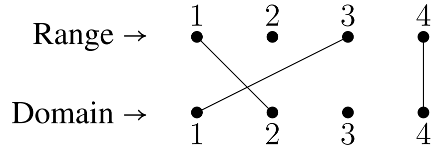
Rook Monoids

The Planar Rook Monoid and its regular representation have been studied in [6]. The goal of this chapter and Chapter 4 is to generalize these results to edge colorings of the Planar Rook Monoid and its associated algebra. This chapter focuses on a Type B analogue of the Planar Rook Monoid, which we realize as $\mathbb{Z}_2$-edge colorings of planar rook diagrams. We will see in Chapter 4 how this is a special case of the G-edge colorings of the Partition Algebra for a group G defined in [1], where in this case $G = \mathbb{Z}_2$.

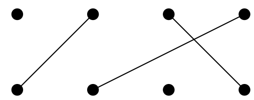
3.1 The Rook Monoid

For each positive integer k, the Rook Monoid R_k can be defined as the set of bijections $d : S \to T$ where S and T are some subsets of $[k] = \{1, 2, \dots, k\}$ with multiplication defined as such: Let $d : S \to T$ and $d' : S' \to T'$ be elements of R_k. Let $I = S \cap T'$, the intersection of the domain of d with the range of d'. Then we define the product of d and d' (in that order) to be the function $d \circ d'$ with domain $(d')^{-1}(I)$ and range $d(I)$ defined by $d \circ d'(s) = d(d'(s))$ for each s in the domain $(d')^{-1}(I)$. We can visualize the element $d : S \to T$ of R_k as a diagram consisting of two rows of k vertices with top row and bottom row labeled $1, 2, \dots, k$ from left to right and edges connecting the vertices in the bottom row corresponding to S to the vertices in the top row corresponding to T.

For example, let $k = 4$ and $d : \{1, 2, 4\} \to \{1, 3, 4\}$ is the function that sends $1 \mapsto 3$, $2 \mapsto 1$ and $4 \mapsto 4$. Then we can visualize this as the diagram:

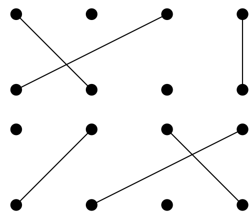


Often, we will draw these diagrams without the numbers. If $d' : \{1, 2, 4\} \to \{2, 3, 4\}$ is the function that sends $1 \mapsto 2$, $2 \mapsto 4$, and $4 \mapsto 3$, this corresponds to the diagram:



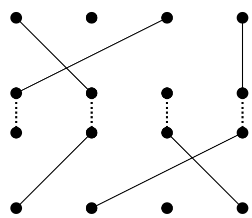
Then to get the product $d \circ d'$, we can first stack d on top of d':

$d =$ $d' =$



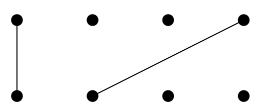
Then we identify the vertices in the two middle rows:

$d =$ $d' =$



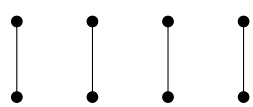
Then for each of the two diagrams, only keep the edges that are incident with an edge in the other diagram. Then we remove the middle row of vertices to create a new diagram, which corresponds to a unique element of R_k. In this case, $d \circ d'$ corresponds to the diagram

$d \circ d' =$



Then the set has an identity, which is the identity map $i : [k] \to [k]$ and corresponds to the diagram with k vertical edges:

$i =$



Therefore, R_k forms a monoid, which is simply a set with a multiplication operation and an identity element for that operation.

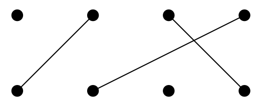
Let the *rank* of an element $d : S \to T$ of R_k, denoted $\mathrm{rk}(d)$, is the size of S (equivalently the size of T or the number of edges in the associated diagram). Note that for $d, d' \in R_k$,

$$\mathrm{rk}(d \circ d') \leq \min\{\mathrm{rk}(d), \mathrm{rk}(d')\}.$$

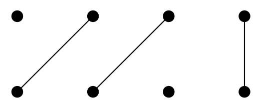
For details on the Rook Monoid's representations and characters see [4] or [11].

3.2 The Planar Rook Monoid

This thesis focuses on the Planar Rook Monoid which we denote PR_k, which is the submonoid of the Rook Monoid consisting of order-preserving functions. These elements correspond to diagrams whose edges do not cross. For example,



is not an element of the Planar Rook Monoid since two of its edges cross, but



is an element of the Planar Rook Monoid since it is an order-preserving function from $\{1,2,4\}$ to $\{2,3,4\}$, meaning that none of its edges cross. It is easy to show that the product of two of these order-preserving functions is again an order-preserving function, so it is closed under the multiplication operation, and PR_k contains the identity element $i : [k] \to [k]$. Therefore, it is indeed a submonoid of R_k.

Note that given two subsets S and T of $[k]$ of the same size, there exists a unique element $d : S \to T$ of PR_k, since d is forced to map the m^{th} largest element of S to the m^{th} largest element of T.

Proposition 3.2.1. *The number of elements in PR_k is $\binom{2k}{k}$.*

Proof. Given an element $d \in PR_k$ with rank $0 \le \ell \le k$, d is completely determined by its domain S and range T. Since there are $\binom{k}{\ell}$ choices for S and $\binom{k}{\ell}$ choices for T, the total number of elements must be

$$\sum_{\ell=0}^{k} \binom{k}{\ell}^2$$

which is well-known to be exactly $\binom{2k}{k}$. We can see this more directly by taking the empty diagram with two rows of k vertices and choosing k of the $2k$ vertices. Then we can define a unique element $d \in PR_k$ with domain equal to the set of chosen vertices in the bottom row of the diagram and range equal to the set of unchosen vertices in the top row. □

Herbig found a presentation of PR_k in [6], which we will use in the following section.

Theorem 3.2.2. *The monoid PR_k has a presentation on generators $\ell_1, \ell_2, \ldots, \ell_{k-1}$, and $r_1, r_2, \ldots, r_{k-1}$ with relations:*

i. $\ell_i^3 = \ell_i^2 = r_i^2 = r_i^3$

ii. *(a)* $r_i r_{i+1} r_i = r_i r_{i+1} = r_{i+1} r_i r_{i+1}$

(b) $\ell_i \ell_{i+1} \ell_i = \ell_i \ell_{i+1} = \ell_{i+1} \ell_i \ell_{i+1}$

iii. *(a)* $r_i \ell_i r_i = r_i$

(b) $\ell_i r_i \ell_i = \ell_i$

iv. *(a)* $r_{i+1} \ell_i r_i = r_{i+1} \ell_i$

(b) $\ell_{i-1} r_i \ell_i = \ell_{i-1} r_i$

(c) $\ell_i r_i \ell_{i+1} = r_i \ell_{i+1}$

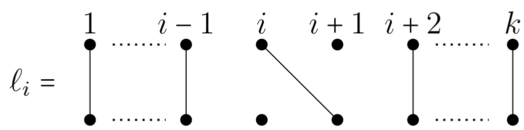
(d) $r_i \ell_i r_{i-1} = \ell_i r_{i-1}$

v. $r_i \ell_i = \ell_{i+1} r_{i+1}$

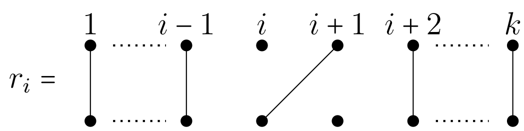
vi. *If* $|i - j| \geq 2$ *then* $r_i \ell_j = \ell_j r_i$, $r_i r_j = r_j r_i$, $\ell_i \ell_j = \ell_j \ell_i$

for all i *and* j *for which each term in the relation is defined.*

For $1 \leq i \leq k - 1$, ℓ_i is associated with the diagram



and r_i is associated with the diagram



3.3 The Planar Rook Monoid of Type B

We define the Planar Rook Monoid of Type B, denoted PR_k^B, to be the set of colored diagrams d_c where $d : S \to T$ is an element of PR_k and $c : S \to \mathbb{Z}_2$ is a coloring of S with $\mathbb{Z}_2$. This is equivalently a coloring of the edges of the diagram associated to d if we say that the coloring of an edge is the coloring of the vertex in S which is incident to that edge. Here we view $\mathbb{Z}_2$ as the multiplicative group on the set $\{1, -1\}$. Then if d_c and $d'_{c'}$ are two elements of PR_k^B, then their product is defined to be

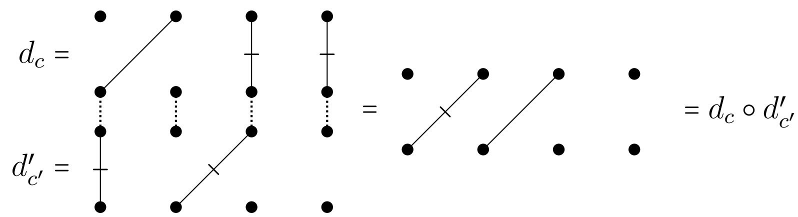
$$d_c \circ d'_{c'} = (d \circ d')_{c''}$$

where $d \circ d' : S \to T$ is the product of d and d' in PR_k and $c'' : S \to \mathbb{Z}_2$ is defined as

$$c''(s) = c(d'(s))c'(s),$$

so the coloring of the edge that is the result of two incident edges in the product of the diagrams is the product of the colorings of those two original edges.

For example, let d_c be an element of PR_k^B with $d : \{1,3,4\} \to \{2,3,4\}$ and $c : \{1,3,4\} \to \mathbb{Z}_2$ defined by $c(1) = 1$, $c(3) = -1$, $c(4) = -1$. Let $d'_{c'}$ be another element with $d' : \{1,2\} \to \{1,3\}$ and $c' : \{1,2\} \to \mathbb{Z}_2$ defined by $c(1) = -1$, $c(2) = -1$. Then $d_c \circ d'_{c'}$ corresponds to the diagram:



Since the left-most edges are colored -1 and $+1$, the color of the resulting edge in the product is -1, and since the color of the next two edges to the right is -1 and -1, the color of the resulting edge in the product is $+1$.

Let the *rank* of an element d_c of PR_k^B be the rank of its underlying diagram d.

Proposition 3.3.1. *The number of elements in* PR_k^B *is* $\sum_{\ell=0}^{k} 2^\ell \binom{k}{\ell}^2$.

Proof. The proof is the same as that for PR_k, except now we have 2^ℓ many colorings for a diagram of rank ℓ, since we can color each edge with either 1 or -1. □

These numbers are called the Central Delannoy Numbers in the literature. For more combinatorial objects counted by the Central Delannoy Numbers and various formulae for Central Delannoy Numbers, see [12].

We will now focus on a presentation of the Planar Rook Monoid of Type B.

Theorem 3.3.2. PR_k^B *has a presentation on generators* $\ell_1, \ell_2, \dots, \ell_{k-1}$, *and* $r_1, r_2, \dots, r_{k-1}$ *and* $p_1, p_2, \dots, p_k$ *with relations 1 – 6 in Theorem 3.2.2, including the following relations:*

vii. $p_i^2 = 1$

viii. *(a)* $p_i r_i = r_i p_{i+1}$

(b) $p_{i+1} \ell_i = \ell_i p_i$

ix. *(a)* $p_i \ell_i = \ell_i = \ell_i p_{i+1}$

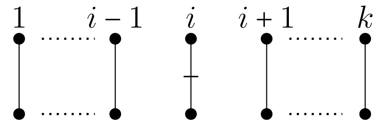
(b) $p_{i+1} r_i = r_i = r_i p_i$

x. *If* $|i - j| \geq 2$ *or* $j = i + 1$, $p_i r_j = r_j p_i$ *and* $p_i \ell_j = \ell_j p_i$

xi. $p_i p_j = p_j p_i$

for all i *and* j *for which each term in the relation is defined.*

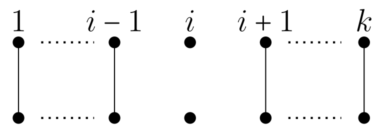
Here, r_i and ℓ_i correspond to the same diagrams as in Theorem 3.2.2 with trivial coloring (every edge colored with +1). The generator p_i corresponds to the diagram:

$p_i =$ 

We will rely on the following Lemma by Herbig.

Lemma 3.3.3 (Herbig [6])**.** *Every element of PR_k is a product of ℓ_i and r_i (or the identity).*

Proof. We will show that every element of PR_k can be written as a word on the letters ℓ_i and r_i. Let e_i be

$e_i =$ 

Then note that

$$e_i = r_i \ell_i, \qquad \text{for } 1 \le i \le k-1$$
$$e_k = \ell_{k-1} r_{k-1}$$

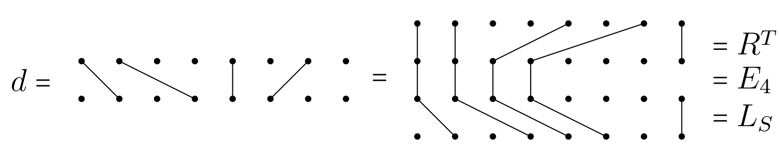
Let $d \in PR_k$ with domain S and range T and rank m. Then let us define $\dot{s} = \max S$ and $\dot{t} = \max T$ and let $\dot{S} = \{\dot{s}+1, \dot{s}+2, \ldots, k\}$ and let $\dot{T} = \{\dot{t}+1, \dot{t}+2, \ldots, k\}$. Then we can decompose d as

$$d = R^T E_m L_S$$

where R^T, E_m and L_S are the diagrams

$$R^T : [m] \cup \dot{T} \to T \cup \dot{T}$$
$$E_m : [m] \to [m]$$
$$L_S : S \cup \dot{S} \to [m] \cup \dot{S}$$

For example, if $k = 8$ and $d : \{2,4,5,6\} \to \{1,2,5,7\}$, then d decomposes as



Let us also define for $1 \le a < b \le k$

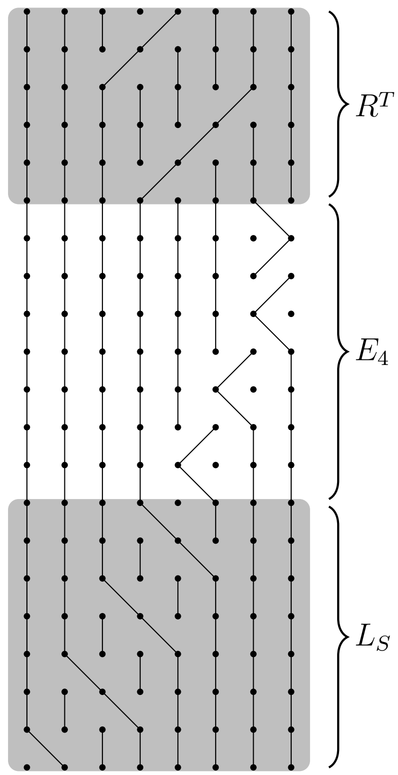
$$R^{a,a} = L_{a,a} = 1$$
$$R^{b,a} = r_{b-1} r_{b-2} \cdots r_a$$
$$L_{a,b} = \ell_a \ell_{a+1} \cdots \ell_{b-1}$$

Then if $S = \{s_1 < s_2 < \cdots < s_m = \dot{s}\}$ and $T = \{t_1 < t_2 < \cdots < t_m = \dot{t}\}$ then we can express R^T, E_m and L_S as

$$\begin{aligned} R^T &= R^{t_1,1} R^{t_2,2} \cdots R^{s_m,m} \\ E_m &= e_{m+1} e_{m+2} \dots e_k \\ L_S &= L_{m,t_m} L_{m-1,t_{m-1}} \cdots L_{1,t_1} \end{aligned}$$

so we see that every element can be written as a word on ℓ_i and r_i in this way. $\square$

Using the same example from our proof, the complete decomposition of $d : \{2,4,5,6\} \to \{1,2,5,7\}$ is:



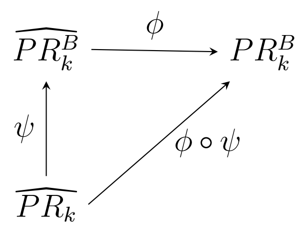
Proof of Theorem 3.3.2. It is easy to see that p_i as shown above along with r_i and ℓ_i as shown earlier satisfy relations 7 – 11. Let $\widehat{PR_k^B}$ be the monoid generated by $\hat{\ell}_i$, $\hat{r}_i$, $\hat{p}_i$ with relations 1 – 11. By Theorem 3.2.2,

$$\widehat{PR}_k := \langle \hat{\ell}_i, \hat{r}_i \rangle \cong PR_k.$$

We also know that PR_k sits in PR_k^B, since the submonoid of PR_k^B with trivial coloring (every edge colored with +1) is isomorphic to PR_k. Let $\psi : \widehat{PR}_k \to \widehat{PR_k^B}$ be the inclusion map and let $\phi : \widehat{PR_k^B} \to PR_k^B$ be the homomorphism defined by

$$\phi(\hat{\ell}_i) = \ell_i$$
$$\phi(\hat{r}_i) = r_i$$
$$\phi(\hat{1}) = 1$$

where $\hat{1}$ is the identity in $\widehat{PR_k^B}$ and 1 is the identity element of PR_k^B, which is the identity map on $[k]$ with trivial coloring. Since the ℓ_i, r_i and p_i satisfy all of the same relations that the $\hat{\ell}_i$, $\hat{r}_i$, and $\hat{p}_i$ do, ϕ must be a well-defined monoid homomorphism, and we have the following commutative diagram:



Suppose that $d_c \in PR_k^B$ with domain T, range S and coloring $c : T \to \mathbb{Z}_2$, then if c_1 is the trivial coloring of d (every edge has color +1), then

$$d_c = \left(\prod_{t \in c^{-1}(-1)} p_t \right) \cdot d_{c_1},$$

(Note: the order of the multiplication does not matter since all p_i commute with each other) but d_{c_1} is in $\psi(\widehat{PR_k})$ so d_{c_1} is a product of ℓ_i and r_i, so d_c is a product of ℓ_i, r_i and p_i. If $d_c = \prod_{j \in J} x_j$ where $x_j \in \{\ell_i, r_i, p_i\}$ then

$$d = \phi\left(\prod_{j \in J} \hat{x}_j \right)$$

where

$$\hat{x}_j = \begin{cases} \hat{\ell}_i & x_j = \ell_i \\ \hat{r}_i & x_j = r_i \\ \hat{p}_i & x_j = p_i \end{cases}$$

so ϕ is surjective.

We call a *standard word* on $\widehat{PR_k^B}$ associated to the diagram $d_c \in PR_k^B$ with $d : S \to T$ and $c : S \to \mathbb{Z}_2$ a word of the form

$$\hat{B}_c \hat{R}^T \hat{E}_m \hat{L}_S$$

where

$$\hat{B}_c = \prod_{t\in c^{-1}(-1)} \hat{p}_t$$

and $\hat{R}^T$, $\hat{E}_m$ and $\hat{L}_S$ are defined analogously as before:

$$\begin{aligned}
\hat{R}^T &= \hat{R}^{t_1,1}\hat{R}^{t_2,2}\dots\hat{R}^{s_m,m}\\
\hat{E}_m &= \hat{e}_{m+1}\hat{e}_{m+2}\dots\hat{e}_k\\
\hat{L}_S &= \hat{L}_{m,t_m}\hat{L}_{m-1,t_{m-1}}\dots\hat{L}_{1,t_1}
\end{aligned}$$

such that

$$\begin{aligned}
\hat{R}^{a,a} &= \hat{L}_{a,a} = \hat{1}\\
\hat{R}^{b,a} &= \hat{r}_{b-1}\hat{r}_{b-2}\dots\hat{r}_a\\
\hat{L}_{a,b} &= \hat{\ell}_a\hat{\ell}_{a+1}\dots\hat{\ell}_{b-1}
\end{aligned}$$

for $1 \le a < b \le k$, and $\hat{e}_i$ is defined as

$$\begin{aligned}
\hat{e}_i &= \hat{r}_i\hat{\ell}_i, \qquad \text{for } 1 \le i \le k-1\\
\hat{e}_k &= \hat{\ell}_{k-1}\hat{r}_{k-1}
\end{aligned}$$

We have shown that ϕ is surjective, so $|\widehat{PR_k^B}| \ge |PR_k^B|$. In order to show that ϕ is an isomorphism, we will show that any element of $\widehat{PR_k^B}$ is equal to a standard word. Since there is exactly one standard word for each diagram in PR_k^B, this shows that $|\widehat{PR_k^B}| = |PR_k^B|$ and hence that ϕ is an isomorphism.

By [6, Theorem 4], we can write any element of $\widehat{PR_k^B}$ with trivial coloring as a standard word with $\hat{B}_c = \hat{1}$.

Let us prove that any word on the letters $\hat{\ell}_i$, $\hat{r}_i$, and $\hat{p}_i$ is equal to a standard word by induction on the length of the word. We know that $\hat{1}$, the unique word of length 0, is trivially a standard word. Suppose any word of length n is equal to a standard word and suppose $\hat{w}$ is a word of length $n+1$. By our inductive hypothesis, the subword of $\hat{w}$ consisting of the last n letters is equal to a standard word, so

$$\hat{w} = x\left(\hat{B}_c\hat{R}^T\hat{E}_m\hat{L}_S\right)$$

for some subsets S and T of $[k]$, $m = |S| = |T|$ and $c : T \to \mathbb{Z}_2$, with $x \in \{\hat{\ell}_i, \hat{r}_i, \hat{p}_i\}$.

Suppose $c^{-1}(-1) = \{t_1, t_2, \dots, t_n\} \subset T$ and $x = r_q$ for some q, then by the relations,

$$\hat{r_q}\hat{p}_{t_1}\hat{p}_{t_2}\dots\hat{p}_{t_n} = \begin{cases} \hat{p}_{t_1-1}\hat{r_q}\hat{p}_{t_2}\dots\hat{p}_{t_n} & t_1 = q+1\\ \hat{r_q}\hat{p}_{t_2}\dots\hat{p}_{t_n} & t_1 = q\\ \hat{p}_{t_1}\hat{r_q}\hat{p}_{t_2}\dots\hat{p}_{t_n} & |t_1 - q| \ge 2 \text{ or } t_1 = q-1 \end{cases}$$

We can continue moving $\hat{r}_q$ right until

$$\hat{r}_q\hat{w} = \hat{p}_{j_1}, \hat{p}_{j_2}, \ldots \hat{p}_{j_{n'}}\hat{r}_q\hat{R}^T\hat{E}_m\hat{L}_S$$

for some $j_1, j_2, \ldots, j_{n'} \le k$. We can allow the j's to be distinct since the $\hat{p}_j$ commute with each other and two $\hat{p}_j$ cancel each other. Since $\hat{r}_q\hat{R}^T\hat{E}_m\hat{L}_S$ is a word on r_i and ℓ_i, it can be rewritten as a standard word $\hat{R}^{T'}\hat{E}_{m'}\hat{L}_{S'}$ so that

$$\hat{r}_q\hat{w} = \hat{p}_{j_1}, \hat{p}_{j_2}, \ldots \hat{p}_{j_{n'}}\hat{R}^{T'}\hat{E}_{m'}\hat{L}_{S'}$$

where $m' = |T'| = |S'|$. Note that we can do the same process for $x = \hat{\ell}_q\hat{w}$ to obtain a word of the same form, and for $x = \hat{p}_q$, this is easy.

In order for this new word to be a standard word, we need the product of the $\hat{p}_{j_i}$ to represent a coloring of the diagram associated to the sets S' and T', so we need each j_i to be in T'. Suppose there is some $\hat{p}_j$ in this word such that $j \notin T'$.

If $j > \max T'$, then if we push $\hat{p}_j$ to the right, it will commute with every r_i in $\hat{R}^{T'}$. By the relations and the definition of $\hat{e}_i$,

$$\hat{p}_j\hat{e}_i = \begin{cases} \hat{e}_i & \text{if } i = j \\ \hat{e}_i\hat{p}_j & \text{if } i \neq j \end{cases}$$

Therefore, we can push $\hat{p}_j$ to the right through $\hat{E}_{m'}$ until we get to $\hat{e}_j$, which will cancel $\hat{p}_j$ and we end up with the same word, minus $\hat{p}_j$.

If $j \le \max T'$, we know that $\hat{p}_j$ commutes with all $\hat{r}_i$ such that $j \neq i$ or $i + 1$. We also know that the first appearance of $\hat{r}_j$ in $\hat{R}^{T'}$ occurs to the left of the first appearance of $\hat{r}_{j-1}$, so we can push $\hat{p}_j$ to the right until we get to $\hat{r}_j$, which will cancel $\hat{p}_j$ and we are left with the same word, minus $\hat{p}_j$. After canceling all of the $j_i \notin T'$, we are left with a standard word, completing our argument that all words on $\hat{\ell}_i$, $\hat{r}_i$ and $\hat{p}_i$ are equal to a standard word. □

Chapter 4

The G-edge Colored Planar Rook Algebra

Just as we have defined a monoid PR_k^B consisting of planar rook diagrams whose edges are colored with the elements of $\mathbb{Z}_2$, we can define a monoid $PR_k(G)$ consisting of diagrams d_c where $d \in PR_k$ and $d : S \to T$, and $c : S \to G$. If $d'_{c'} \in PR_k(G)$ as well, then we define the product

$$d_c \circ d'_{c'} = (d \circ d')_{c''}$$

where $d \circ d' : S \to T$ is the product of d and d' in PR_k and $c'' : S \to G$ is defined as

$$c''(s) = c(d'(s))c'(s),$$

Note that the order of multiplication of these two elements in G is important since we are talking about any group, not necessarily abelian. We also see that this set has an identity, which is the identity diagram $d : [k] \to [k]$ with trivial coloring (the map which takes everything to the identity of G). Let us denote the set of colorings c (for a fixed group G) of d by $\text{Col}(d)$,

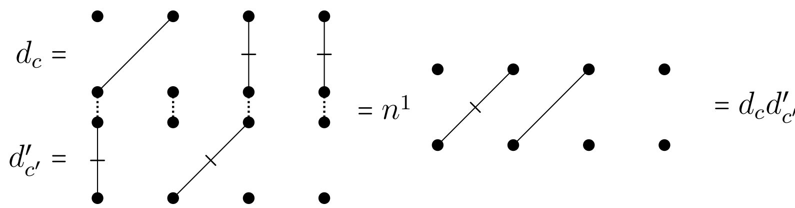
$$\text{Col}(d) := \{c \,|\, c : S \to G\}$$

From the monoid PR_k^B, we can obtain a collection of algebras $PR_k(n; G)$ with $n \in \mathbb{C}\backslash\{0\}$. Let $PR_k(n; G) = \mathbb{C}PR_k(G)$, the $\mathbb{C}-$span of the set of G-edge colored planar rook diagrams, and let the product of two diagrams d_c and $d'_{c'}$ in the algebra (which we write as juxtaposition of the two elements) be defined

$$d_c d'_{c'} := n^{\sigma}(d_c \circ d'_{c'})$$

where if $d : S \to T$ and $d' : S' \to T'$, $\sigma = k - |T \cup S'|$, the number of vertices in the middle row which are not incident to any edge during the composition operation of the two diagrams d and d', and $d_c \circ d'_{c'}$ is again the product of the diagrams in the underlying monoid $PR_k(G)$.

For example, in the algebra $PR_k(n; \mathbb{Z}_2)$, the elements are generated by the Type B planar rook diagrams and



since the number of vertices in the middle row (after identification) that are not incident to any edge is 1. We have defined these algebras to be subalgebras of the G-edge Colored Partition Algebra $P_k(n;G)$. For more information on the G-edge Colored Partition Algebra and its representation-theoretic importance, see [1].

4.1 $PR_k(n;G)$ for Finite Abelian Groups

Let us now study $PR_k(n;G)$ with G be a finite abelian group. We will decompose its regular representation into a direct sum of irreducible represenations, determine which of these irreducible representations are distinct, and determine how its irreducible representations restrict to representations of $PR_{k-1}(n;G)$. We will view G as the direct sum of cyclic groups

$$G = C_{q_1} \oplus C_{q_2} \oplus \cdots \oplus C_{q_m}$$

where we view C_q as the additive group structure on the set $\{0, 1, \ldots, q-1\}$ modulo q. Then a typical element $g \in G$ is

$$g = (g_1, g_2, \ldots, g_m)$$

where $0 \le g_i < q_i$ for each i.

4.1.1 Regular Representation

Recall that the regular representation of $PR_k(n;G)$ is the representation of $PR_k(n;G)$ over itself, and the action of an element in $PR_k(n;G)$ on an element of the representation $PR_k(n;G)$ is just defined by multiplication from the left in the algebra.

Definition 4.1.1. Given a group G, call a G-partition of $[k]$ a set

$$A = \{A^g : g \in G\}$$

of pairwise disjoint subsets A^g of $[k]$ (blocks) indexed by the group elements $g \in G$ (Note: The blocks in A may not partition the entire set $[k]$, i.e. $\bigcup A = \bigcup_{g \in G} A^g$ may be a proper subset of $[k]$).

If A is a G-partition of $[k]$, then for any $c : S \to G$ such that $\bigcup A \subseteq S$, define the complex number

$$\alpha(A, c) = \prod_{g \in G} \prod_{i \in A^g} \prod_{1 \le j \le m} (\zeta_{q_j})^{g_j c(i)_j} \tag{4.1}$$

where ζ_{q_j} is the root of unity $e^{2\pi i/q_j}$ and the empty product is defined naturally as 1.

Definition 4.1.2. If A is a G-partition of $[k]$ and $d \in PR_k$ with $d : S \to T$ such that $\bigcup A \subseteq S$, then define

$$d(A) := \{d(A^g) \,:\, g \in G\},$$

the G-partition of $[k]$ where the block indexed by $g \in G$ is $d(A^g)$.

Also, let us denote the domain of a diagram $d \in PR_k$ as $\mathrm{dom}(d)$ and the range (equivalently the image) of d as $\mathrm{img}(d)$

Lemma 4.1.3. *Let $d_c \in PR_k(G)$, and let $d'_{c'} \in PR_k(G)$ such that $\mathrm{img}(d') \subseteq \mathrm{dom}(d)$.*

i. If $d_c \circ d'_{c'} = d''_{c''}$ then for any G-partition A of $[k]$ we have

$$\alpha(A, c'') = \alpha(d'(A), c)\,\alpha(A, c').$$

ii. If A_1 and A_2 are G-partitions of $[k]$ such that $\bigcup A_1$ and $\bigcup A_2$ are disjoint and both contained in $\mathrm{dom}(d)$, then

$$\alpha(A_1 \cup A_2, c) = \alpha(A_1, c)\,\alpha(A_2, c)$$

where $A_1 \cup A_2$ is the G-partition of $[k]$ where the block indexed by $g \in G$ is $A_1^g \cup A_2^g$.

Proof. These are both immediate from the definition of α and the fact that G is abelian. □

Given A_1 and A_2, G-partitions of $[k]$, define a partial relation $\le$ on these set partitions where

$$A_1 \le A_2 \text{ iff } A_1^g = A_2^g \text{ for all } g \ne 0 \text{ and } A_1^0 \subseteq A_2^0.$$

where 0 is the identity element $(0, 0, \ldots, 0) \in G$. If $T \subseteq [k]$ such that $|T| = |\bigcup A|$, we can define the element $y_A^T \in PR_k(n; G)$ in terms of the α function:

$$y_A^T = \sum_{A_1 \le A} \left(\frac{-|G|}{n}\right)^{|A^0 \setminus A_1^0|} \sum_{c \in \mathrm{Col}(d|_{\bigcup A_1})} \alpha(A_1, c)\,(d|_{\bigcup A_1})_c \tag{4.2}$$

where $d|_{\bigcup A_1}$ is the diagram d with domain restricted to $\bigcup A_1$. Note that the colorings of the edges coming from $\bigcup_{g \ne 0} A^g$ contribute to the coefficient of that diagram in the sum, but not A^0. Let us consider the span over $\mathbb{C}$ of all y_A^T for a fixed G-partition A:

$$Y_A^k = \mathrm{span}_{\mathbb{C}}\{y_A^T : T \subseteq [k] \text{ and } |T| = |\bigcup A|\}.$$

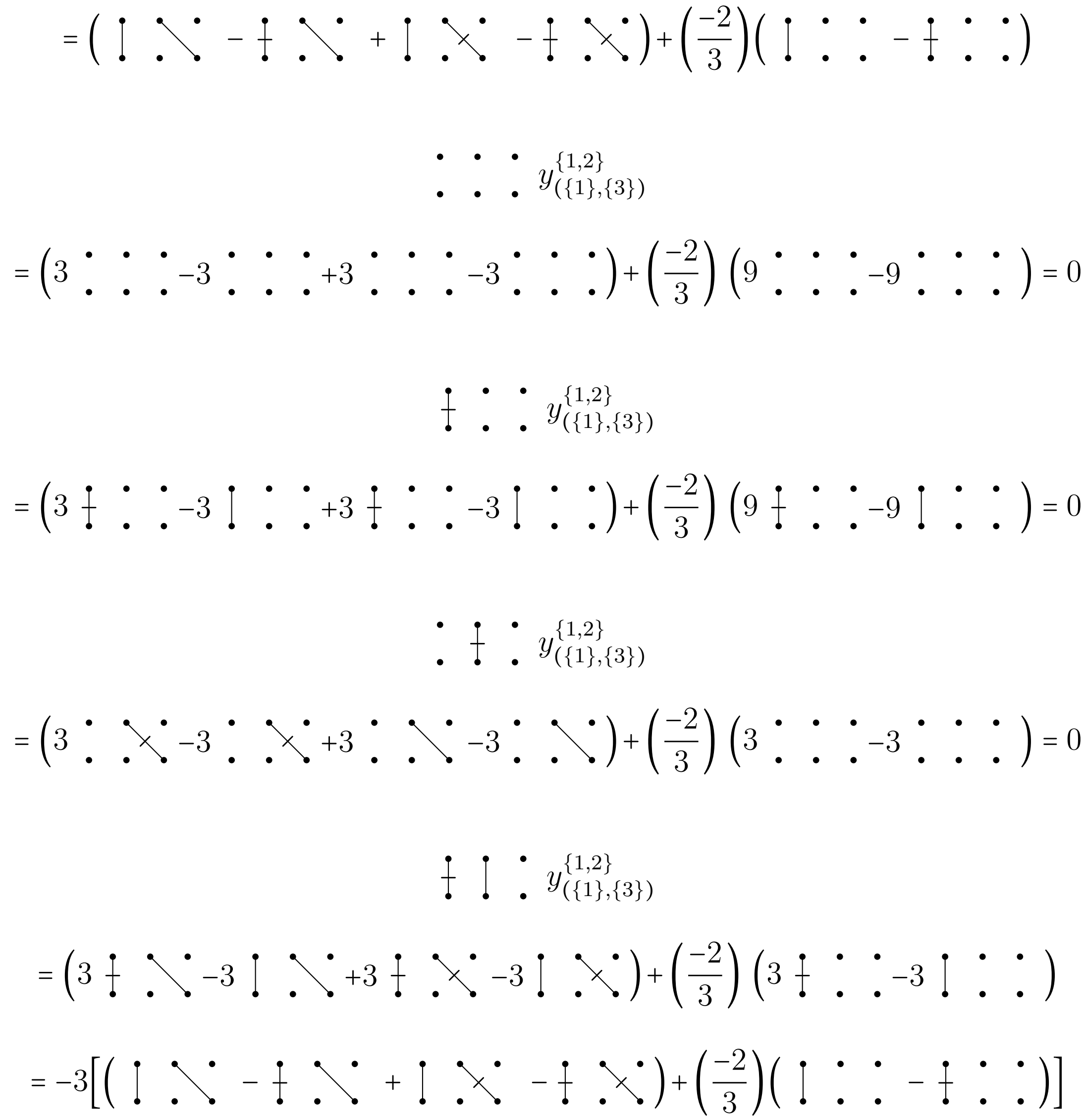
Proposition 4.1.4. *Let $d'_{c'} \in PR_k(G)$ and $d : \bigcup A \to T$. Then the product of $d'_{c'}$ with y_A^T in $PR_k(n; G)$ has the form*

$$d'_{c'} y_A^T = \begin{cases} n^{k - \mathrm{rk}(d')}\,\alpha(d(A), c')^{-1}\,y_A^{d'(T)} & \text{if } T \subseteq \mathrm{dom}(d') \\ 0 & \text{if } T \nsubseteq \mathrm{dom}(d') \end{cases}$$

Before we prove this proposition, let's look at an example of this multiplication.

Example 4.1.5. Let $n = k = 3$ and let $G = \mathbb{Z}_2$ be the additive group on $\{0, 1\}$ modulo 2 this time (before we were thinking of $\mathbb{Z}_2$ as the multiplicative group on $\{\pm 1\}$). Then let us look at the actions of various colored diagrams on y_A^T where $A^0 = \{3\}$ and $A^1 = \{1\}$ and $T = \{1, 2\}$ (now we represent edges colored with 1 by tick marks and edges colored by 0 without tick marks).

$$y^{\{1,2\}}_{(\{1\},\{3\})} = \sum_{A_1^+ \subseteq \{3\}} \left(\frac{-2}{3}\right)^{|\{3\} \setminus A_1^+|} \sum_{\substack{c \text{ coloring} \\ \text{of } d|_{\{1\} \cup A_1^+}}} \alpha_{\{1\},c}(d|_{\{1\} \cup A_1^+})_c$$



$$= -3y^{\{1,2\}}_{(\{1\},\{3\})}$$

In order to prove Proposition 4.1.4, let us first write each y_A^T element in terms of other elements in Y_A^k.

Claim 4.1.6.

$$y_A^T = \sum_{c\in \mathrm{Col}(d)} \alpha(A,c)d_c - \sum_{A_1<A} \left(\frac{|G|}{n}\right)^{|A^0\setminus A_1^0|} y_{A_1}^{d(\bigcup A_1)}$$

Proof. Inserting the definition of $y_{A_1}^{d(\bigcup A_1)}$ into the right side gives us

$$= \sum_{c\in \mathrm{Col}(d)} \alpha(A,c)d_c - \sum_{A_1<A}(-1)^{|A^0\setminus A_1^0|}\left(-\frac{|G|}{n}\right)^{|A^0\setminus A_1^0|}\left(\sum_{A_2\le A_1}\left(-\frac{|G|}{n}\right)^{|A_1^0\setminus A_2^0|}\sum_{c\in \mathrm{Col}(d|_{\bigcup A_2})}\alpha(A_2,c)(d|_{\bigcup A_2})_c\right)$$

$$= \sum_{c\in \mathrm{Col}(d)} \alpha(A,c)d_c - \sum_{A_1<A}\sum_{A_2\le A_1}(-1)^{|A^0\setminus A_1^0|}\left(-\frac{|G|}{n}\right)^{|A^0\setminus A_2^0|}\sum_{c\in \mathrm{Col}(d|_{\bigcup A_2})}\alpha(A_2,c)(d|_{\bigcup A_2})_c$$

For a given $A_2 < A$, the coefficient of

$$\left(-\frac{|G|}{n}\right)^{|A^0\setminus A_2^0|}\sum_{c\in \mathrm{Col}(d|_{\bigcup A_2})}\alpha(A_2,c)(d|_{\bigcup A_2})_c$$

is going to be the sum over all ℓ of the number of A_1 such that $A_2 \le A_1 < A$ and $\ell = |A_1^0\setminus A_2^0|$. Since there are $\binom{|A^0\setminus A_2^0|}{\ell}$ many of these, we have

$$-\sum_{\ell=0}^{|A^0\setminus A_2^0|-1}(-1)^{|A^0\setminus A_2^0|-\ell}\binom{|A^0\setminus A_2^0|}{\ell}$$
$$= -\sum_{\ell=1}^{|A^0\setminus A_2^0|}(-1)^{\ell}\binom{|A^0\setminus A_2^0|}{\ell} = -(0-1) = 1$$

so our formula gives

$$y_A^T = \sum_{c\in \mathrm{Col}(d)} \alpha(A,c)d_c - \sum_{A_2<A}\left(-\frac{|G|}{n}\right)^{|A^0\setminus A_2^0|}\sum_{c\in \mathrm{Col}(d|_{\bigcup A_2})}\alpha(A_2,c)(d|_{\bigcup A_2})_c$$

which exactly agrees with our original definition of y_A^T. □

Claim 4.1.7. *Given $d'_{c'} \in PR_k(G)$, $d \in PR_k$, and A a G-partition of $[k]$, if $d(\bigcup_{g\neq 0} A^g)\setminus \mathrm{dom}(d') \neq \varnothing$ then*

$$\sum_{c\in \mathrm{Col}(d)} \alpha(A,c)d'_{c'}d_c = 0$$

Proof. Suppose $s \in A^{\widehat{g}}$ for some $\widehat{g} \neq 0$ such that $d(s) \notin \mathrm{dom}(d')$. For each $c' \in \mathrm{Col}(d|_{(\bigcup A)\setminus\{s\}})$, we can break up the sum into the smaller sums

$$\sum_{\substack{c\in\mathrm{Col}(d)\\ c|_{S\setminus\{s\}}=c'}} \alpha(A,c)d'_{c'}d_c = n^{k-|T\cup\mathrm{dom}(d')|}\left(\sum_{\substack{c\in\mathrm{Col}(d)\\ c|_{S\setminus\{s\}}=c'}} \alpha(A,c)\right)(d'_{c'} \circ (d|_{(\bigcup A)\setminus\{s\}})_{c'})$$

but now

$$\sum_{\substack{c\in\mathrm{Col}(d)\\ c|_{S\setminus\{s\}}=c'}} \alpha(A,c) = \left(\prod_{g\in G}\prod_{s\neq i\in A^g}\prod_{1\leq j\leq m}(\zeta_{q_j})^{g_j c'(i)_j}\right)\sum_{\substack{c\in\mathrm{Col}(d)\\ c|_{S\setminus\{s\}}=c'}}\prod_{1\leq j\leq m}(\zeta_{q_j})^{\widehat{g}_j c(s)_j}$$

but we have

$$\sum_{\substack{c\in\mathrm{Col}(d)\\ c|_{S\setminus\{s\}}=c'}}\prod_{1\leq j\leq m}(\zeta_{q_j})^{\widehat{g}_j c(s)_j} = \prod_{1\leq j\leq m}\left(\sum_{g_j=0}^{q_j-1}((\zeta_{q_j})^{\widehat{g}_j})^{g_j}\right).$$

Since $\widehat{g} \neq 0$, there must be some j such that $\widehat{g}_j \neq 0$, so for this j

$$\sum_{g_j=0}^{q_j-1}((\zeta_{q_j})^{\widehat{g}_j})^{g_j} = \left(\frac{(\zeta_{q_j})^{\widehat{g}_j q_j}-1}{(\zeta_{q_j})^{\widehat{g}_j}-1}\right) = \left(\frac{1-1}{(\zeta_{q_j})^{\widehat{g}_j}-1}\right) = 0$$

so the whole product above must be 0, thus the whole sum must be 0. □

Proof of Proposition 4.1.4. Now that we have a recursive formula for the basis vectors, we can apply induction on the size of A^0.

Base Case: As our base case, let A be such that $A^0 = \varnothing$. By definition,

$$y_A^T = \sum_{c\in\mathrm{Col}(d)} \alpha(A,c)d_c$$

so

$$d'_{c'}y_A^T = \sum_{c\in\mathrm{Col}(d)} \alpha(A,c)d'_{c'}d_c.$$

Suppose that $T \nsubseteq \mathrm{dom}(d')$, then since $A^0 = \varnothing$, $d(\bigcup_{g\neq 0} A^g)\setminus\mathrm{dom}(d') \neq \varnothing$, so by Claim 4.1.7 this sum is 0.

Suppose that instead $T \subseteq \mathrm{dom}(d')$. Then for every $c'' \in \mathrm{Col}(d' \circ d)$, $(d' \circ d)_{c''} = d'_{c'} \circ d_c$ for a unique $c \in \mathrm{Col}(d)$, so using the formula in Claim 4.1.3 we have

$$\begin{aligned}
d'_{c'}y_A^T &= \sum_{c\in\mathrm{Col}(d)} \alpha(A,c)d'_{c'}d_c \\
&= n^{k-\mathrm{rk}(d')}\alpha(d(A),c')^{-1}\sum_{c\in\mathrm{Col}(d)}\alpha(d(A),c')\alpha(A,c)(d'_{c'}\circ d_c) \\
&= n^{k-\mathrm{rk}(d')}\alpha(d(A),c')^{-1}\sum_{c''\in\mathrm{Col}(d'\circ d)}\alpha(A,c'')(d'\circ d)_{c''} \\
&= n^{k-\mathrm{rk}(d')}\alpha(d(A),c')^{-1}y_A^{d'(T)}.
\end{aligned}$$

Inductive Step: Assume that our hypothesis is true for all $d'_{c'}$ and y_A^T with $|A^0| < N$ for some integer $N > 0$. Suppose we have y_A^T with $|A^0| = N$.

$$d'_{c'} y_A^T = \sum_{c \in \mathrm{Col}(d)} \alpha(A,c) d'_{c'} d_c - \sum_{A_1 < A} \left(\frac{|G|}{n}\right)^{|A^0 \setminus A_1^0|} d'_{c'} y_{A_1}^{d(\cup A_1)}$$

Let $T^0 := d(A^0)$ and let $T^+ = d(\bigcup_{g \neq 0} A^g)$. Then we have three cases:

i. $|T^0 \setminus \mathrm{dom}(d')| = 0$ and $|T^+ \setminus \mathrm{dom}(d')| = 0$

ii. $|T^0 \setminus \mathrm{dom}(d')| > 0$ and $|T^+ \setminus \mathrm{dom}(d')| = 0$

iii. $|T^+ \setminus \mathrm{dom}(d')| > 0$

Case 1: If $|T^0 \setminus \mathrm{dom}(d')| = 0$ and $|T^+ \setminus \mathrm{dom}(d')| = 0$ then $T \subseteq \mathrm{dom}(d')$ and

$$\begin{aligned} d'_{c'} y_A^T = & \sum_{c \in \mathrm{Col}(d)} \alpha(A,c) d'_{c'} d_c - \sum_{A_1 < A} \left(\frac{|G|}{n}\right)^{|A^0 \setminus A_1^0|} d'_{c'} y_{A_1}^{d(\cup A_1)} \\ = & n^{k - \mathrm{rk}(d')} \alpha(d(A), c')^{-1} \sum_{c'' \in \mathrm{Col}(d' \circ d)} \alpha(A, c'') (d' \circ d)_{c''} \\ & - \sum_{A_1 < A} \left(\frac{|G|}{n}\right)^{|A^0 \setminus A_1^0|} n^{k - \mathrm{rk}(d')} \alpha(d(A_1), c')^{-1} y_{A_1}^{d' \circ d(\cup A_1)} \end{aligned}$$

but $\alpha(d(A_1), c') = \alpha(d(A), c')$ for all $A_1 < A$ since $d(A^g) = d(A_1^g)$ for all $g \neq 0$ and $d(A^0)$ does not contribute to the α coefficient, so this is

$$\begin{aligned} = & n^{k - \mathrm{rk}(d')} \alpha(d(A), c')^{-1} \left(\sum_{c'' \in \mathrm{Col}(d' \circ d)} \alpha(A, c'') (d' \circ d)_{c''} - \sum_{A_1 < A} \left(\frac{|G|}{n}\right)^{|A^0 \setminus A_1^0|} y_{A_1}^{d' \circ d(\cup A_1)} \right) \\ = & n^{k - \mathrm{rk}(d')} \alpha(d(A), c')^{-1} y_A^{d'(T)} \end{aligned}$$

Case 2: If $|T^0 \setminus \mathrm{dom}(d')| > 0$ and $|T^+ \setminus \mathrm{dom}(d')| = 0$, for each $c'' \in \mathrm{Col}(d' \circ d)$ there are $|G|^{|T^0 \setminus \mathrm{dom}(d')|}$ many choices for $c \in \mathrm{Col}(d)$ such that $d'_{c'} \circ d_c = (d' \circ d)_{c''}$. Also,

$$d'_{c'} d_c = n^{k - |T^0 \cup \mathrm{dom}(d')|} d'_{c'} \circ d_c = n^{k - \mathrm{rk}(d') - |T^0 \setminus \mathrm{dom}(d')|} d'_{c'} \circ d_c.$$

Therefore, if we let $A_0 < A$ be the G-partition of $[k]$ such that $A_0^0 = d^{-1}(T^0 \cap \mathrm{dom}(d'))$,

$$\sum_{c \in \mathrm{Col}(d)} \alpha(A,c) d'_{c'} d_c = \left(\frac{|G|}{n}\right)^{|T^0 \setminus \mathrm{dom}(d')|} n^{k - \mathrm{rk}(d')} \alpha(d(A), c')^{-1} \sum_{c'' \in \mathrm{Col}(d' \circ d)} \alpha(A_0, c'') (d' \circ d)_{c''}$$

By our inductive hypothesis and the fact that $T^+ \subseteq \operatorname{dom}(d')$, $d'_{c'} y_{A_1}^{d(\bigcup A_1)} = 0$ if $A_1^0 \notin d^{-1}(T^0 \cap \operatorname{dom}(d'))$, so

$$\begin{aligned}
&\sum_{A_1 < A} \left(\frac{|G|}{n}\right)^{|A^0 \setminus A_1^0|} d'_{c'}\, y_{A_1}^{d(\bigcup A_1)} \\
&= \sum_{A_1 \le A_0} \left(\frac{|G|}{n}\right)^{|A^0 \setminus A_1^0|} d'_{c'}\, y_{A_1}^{d(\bigcup A_1)} \\
&= \sum_{A_1 \le A_0} \left(\frac{|G|}{n}\right)^{|A^0 \setminus A_1^0|} n^{k - \operatorname{rk}(d')}\, \alpha(d(A_1), c')^{-1}\, y_{A_1}^{d' \circ d(\bigcup A_1)}
\end{aligned}$$

but $\alpha(d(A_1), c') = \alpha(d(A), c')$ like before, so we need to show that

$$\left(\frac{|G|}{n}\right)^{|T^0 \setminus \operatorname{dom}(d')|} \sum_{c'' \in \operatorname{Col}(d' \circ d)} \alpha(A, c'')(d' \circ d)_{c''} = \sum_{A_1 \le A_0} \left(\frac{|G|}{n}\right)^{|A^0 \setminus A_1^0|} y_{A_1}^{d' \circ d(\bigcup A_1)}$$

The right side is equal to:

$$\begin{aligned}
&\sum_{A_1 \le A_0} \left(\frac{|G|}{n}\right)^{|A^0 \setminus A_1^0|} y_{A_1}^{d' \circ d(\bigcup A_1)} \\
&= \sum_{A_1 \le A_0} \left(\frac{|G|}{n}\right)^{|A^0 \setminus A_1^0|} \left(\sum_{A_2 < A_1} \left(\frac{-|G|}{n}\right)^{|A_1^0 \setminus A_2^0|} \sum_{c \in \operatorname{Col}(d' \circ d|_{\bigcup A_2})} \alpha(A_2, c)(d' \circ d|_{\bigcup A_2})_c \right) \\
&= \sum_{A_1 \le A_0} \sum_{A_2 < A_1} (-1)^{|A_1^0 \setminus A_2^0|} \left(\frac{|G|}{n}\right)^{|A^0 \setminus A_2^0|} \sum_{c \in \operatorname{Col}(d' \circ d|_{\bigcup A_2})} \alpha(A_2, c)(d' \circ d|_{\bigcup A_2})_c
\end{aligned}$$

For all $A_2 < A_0$, the coefficient of

$$\sum_{c \in \operatorname{Col}(d' \circ d|_{\bigcup A_2})} \alpha(A_2, c)(d' \circ d|_{\bigcup A_2})_c$$

in this sum is

$$\sum_{\ell=0}^{|T^0 \cap \operatorname{dom}(d')| - |A_2^0|} (-1)^\ell \left(\frac{|G|}{n}\right)^{|A^0 \setminus A_2^0|} \binom{|T^0 \cap \operatorname{dom}(d')| - |A_2^0|}{\ell}$$

$$= \left(\frac{|G|}{n}\right)^{|A^0 \setminus A_2^0|} (1-1)^{|T^0 \cap \operatorname{dom}(d')| - |A_2^0|} = 0.$$

For $A_2 = A_0$, the coefficient of

$$\sum_{c \in \operatorname{Col}(d' \circ d|_{\bigcup A_0})} \alpha(A_0, c)(d' \circ d|_{\bigcup A_0})_c = \sum_{c'' \in \operatorname{Col}(d' \circ d)} \alpha(A, c'')(d' \circ d)_{c''}$$

is
$$\left(\frac{|G|}{n}\right)^{|A^0\backslash A_0^0|}=\left(\frac{|G|}{n}\right)^{|T^0\backslash d(A_0^0)|}=\left(\frac{|G|}{n}\right)^{|T^0\backslash (T^0\cap \mathrm{dom}(d'))|}=\left(\frac{|G|}{n}\right)^{|T^0\backslash \mathrm{dom}(d')|}$$

Case 3: Finally, if $|T^+\backslash \mathrm{dom}(d')|>0$ then by Claim 4.1.7,
$$\sum_{c\in \mathrm{Col}(d)}\alpha(A,c)d'_{c'}d_c=0.$$

For all $A_1<A$, since $T^+\subseteq d(\bigcup A_1)$ and $T^+\nsubseteq \mathrm{dom}(d')$, then $d(\bigcup A_1)\nsubseteq \mathrm{dom}(d')$. Then by our inductive hypothesis
$$d'_{c'}y_{A_1}^{d(\bigcup A_1)}=0$$
so the whole sum is 0.

Therefore, the claim follows by induction. □

Note that this multiplication in the algebra is exactly the action in the regular representation. We want to show that the y_A^T form a basis of the algebra. First we start with a lemma and then prove that they do form a basis.

Lemma 4.1.8. *Let $d\in PR_k$ with $d:S\to T$ and $\mathrm{rk}(d)=\ell$. For any colorings c and c' of d,*
$$\sum_{A:\bigcup A=S}\alpha(A,c)^{-1}\alpha(A,c')=\begin{cases}|G|^\ell & \text{if } c=c'\\ 0 & \text{if } c\neq c'\end{cases}\tag{4.3}$$

Proof. If $c=c'$ then $\alpha(A,c)^{-1}\alpha(A,c')=1$. Since the number of colorings of d is $|G|^\ell$, the sum must be $|G|^\ell$.

If $c\neq c'$, then let $s\in S$ such that $c(s)\neq c'(s)$. For each A, a G-partition of $[k]$ such that $\bigcup A=S\backslash\{s\}$, then for $g\in G$ let A_g be the G-partition of $[k]$ such that $A_g^{g'}=A^{g'}$ for all $g'\neq g$ and $A_g^g=A^g\cup\{s\}$. Then
$$\begin{aligned}\sum_{g\in G}\alpha(A_g,c)^{-1}\alpha(A_g,c')&=\alpha(A,c)^{-1}\alpha(A,c')\sum_{g\in G}\left(\prod_{1\le j\le m}(\zeta_{q_j})^{g_jc(s)_j}\right)^{-1}\left(\prod_{1\le j\le m}(\zeta_{q_j})^{g_jc'(s)_j}\right)\\&=\alpha(A,c)^{-1}\alpha(A,c')\sum_{g\in G}\prod_{1\le j\le m}\left((\zeta_{q_j})^{c'(s)_j-c(s)_j}\right)^{g_j}\\&=\alpha(A,c)^{-1}\alpha(A,c')\prod_{1\le j\le m}\left(\sum_{g_j=0}^{q_j-1}\left((\zeta_{q_j})^{c'(s)_j-c(s)_j}\right)^{g_j}\right)\end{aligned}$$

Since $c(s)\neq c'(s)$, there is some j such that $c(s)_j\neq c'(s)_j$ so
$$\sum_{g_j=0}^{q_j-1}\left((\zeta_{q_j})^{c'(s)_j-c(s)_j}\right)^{g_j}=\frac{\left((\zeta_{q_j})^{c'(s)_j-c(s)_j}\right)^{q_j}-1}{(\zeta_{q_j})^{c'(s)_j-c(s)_j}-1}=\frac{1-1}{(\zeta_{q_j})^{c'(s)_j-c(s)_j}-1}=0,$$
hence (4.3) sums to 0. □

Proposition 4.1.9. *The set of y_A^T where A is a G-partition of $[k]$ forms a basis of $PR_k(n;G)$.*

Proof. We know that

$$\#\{y_A^T\} = \sum_{\ell=0}^{k} |G|^\ell \binom{k}{\ell}^2 = \dim PR_k^B(n)$$

since we can create a one-to-one correspondence between the elements y_A^T and colored diagrams by associating the element y_A^T with the diagram with domain $\bigcup A$ and range T, where we color the edge incident to s in the domain with the group element corresponding to the block in A containing s.

Then it is enough to show that every colored diagram $d_c \in PR_k^B$ is in

$$Y := \sum_A Y_A^k = \mathrm{span}_{\mathbb{C}}\{y_A^T : A \text{ a } G\text{-partition}, T \subseteq [k], |T| = |\bigcup A|\}$$

since the colored diagrams d_c form a basis of $PR_k^B(n)$, by construction. We argue by induction on rank:

Base Case: For $d_c \in PR_k(n;G)$ with $\mathrm{rk}(d) = 0$, $d_c = y_{A_\varnothing}^\varnothing$ the empty diagram, where $A_\varnothing$ is the G-partition such that $A_\varnothing^g = \varnothing$ for each $g \in G$.

Inductive Step: Assume inductively that for $\ell > 0$, all colored diagrams of rank less than ℓ are in Y. Recall that by Claim 4.1.6 we can rewrite y_A^T recursively as:

$$y_A^T = \sum_{c \in \mathrm{Col}(d)} \alpha(A,c) d_c - \sum_{A_1 < A} \left(\frac{|G|}{n}\right)^{|A^0 \setminus A_1^0|} y_{A_1}^{d(\bigcup A_1)} \tag{4.4}$$

Let $d_c \in PR_k(n;G)$ with $d : S \to T$ and $\mathrm{rk}(d) = \ell$ (Note that c is a fixed coloring of the diagram d now), and let

$$y_{d_c} = \sum_{A \text{ s.t. } \bigcup A = S} \alpha(A,c)^{-1}\, y_A^T$$

but by (4.4), this is equal to

$$= \sum_{A \text{ s.t. } \bigcup A = S} \alpha(A,c)^{-1} \left(\sum_{c' \in \mathrm{Col}(d)} \alpha(A,c') d_{c'} - \sum_{A_1 < A} \left(\frac{|G|}{n}\right)^{|A^0 \setminus A_1^0|} y_{A_1}^{d(\bigcup A_1)} \right)$$

$$= \sum_{A \text{ s.t. } \bigcup A = S} \sum_{c' \in \mathrm{Col}(d)} \alpha(A,c)^{-1} \alpha(A,c') d_{c'} - \sum_{A \text{ s.t. } \bigcup A = S} \sum_{A_1 < A} \left(\frac{|G|}{n}\right)^{|A^0 \setminus A_1^0|} \alpha(A,c)^{-1} y_{A_1}^{d(\bigcup A_1)}$$

The coefficient of $d_{c'}$ in y_{d_c} is exactly the expression in (4.3), which is nonzero if and only if $c = c'$. Therefore, the first sum evaluates to $|G|^\ell d_c$ and the rank of all of the diagrams in the sum $y_{d_c} - |G|^\ell d_c$ must be less than ℓ. By our inductive assumption, each of those diagrams is in Y, so $y_{d_c} - |G|^\ell d_c \in Y$ hence $y_{d_c} \in Y$. □

Theorem 4.1.10. *Each Y_A^k is an irreducible subrepresentation of the regular representation on $PR_k(n;G)$.*

Proof. By Proposition 4.1.4, the action of an element of $PR_k(n;G)$ on a basis element is either a constant multiple of another basis element or 0, so Y_A^k is a subrepresentation of the regular representation.

Furthermore, given y_A^T and $y_A^{T'}$ basis elements in Y_A^k, let $d : T \to T'$ and let $c_1 : S \to G$ be the trivial coloring, i.e. c_1 is the trivial map $c_1(s) = 0$ for all $s \in S$. Then $d_{c_1}\, y_A^T = n^{k-|T|}\, y_A^{d(T)} = n^{k-|T|}\, y_A^{T'}$, so every basis element generates Y_A^k.

Let

$$y = \sum_{T \text{ s.t. } |T| = |\bigcup A|} \lambda_T\, y_A^T \neq 0.$$

Then there exists some T' such that $\lambda_{T'} \neq 0$, so if we let $d : T' \to T'$ and c_1 be the trivial coloring, then

$$d_{c_1}\, y = \sum_{T \text{ s.t. } |T| = |\bigcup A|} \lambda_T\, d_{c_1}\, y_A^T = n^{k-|T'|} \lambda_{T'}\, y_A^{T'}$$

since the only subset with size $|T'|$ such that it is contained in T' is T' itself, and this element generates Y_A^k, so every element generates the whole space. Therefore, Y_A^k is irreducible. □

Therefore, we have completely decomposed the regular representation into a direct sum of irreducible representations, and the following theorem is immediate.

Theorem 4.1.11. *The algebra $PR_k(n;G)$ decomposes in the following way into a direct sum of irreducible sub-representations of its regular representation*

$$PR_k(n;G) = \bigoplus_{A \text{ a } G\text{-partition of } [k]} Y_A^k$$

Proposition 4.1.12. *The algebra $PR_k(n;G)$ is semisimple and any finite dimensional irreducible representation of $PR_k(n;G)$ is isomorphic to Y_A^k for some G-partition of $[k]$.*

Proof. By Theorem 4.1.11, the algebra's regular representation is completely reducible. By Proposition 2.1.2, the algebra is then semisimple. □

Now that we know what all of the finite dimensional irreducible representations look like, let us now look at which ones are distinct.

Proposition 4.1.13. $Y_{A_1}^k \cong Y_{A_2}^k$ *as representations of $PR_k(n;G)$ if and only if $|A_1^g| = |A_2^g|$ for all $g \in G$, $\bigcup A_1 = \{x_1 < x_2 < \cdots < x_p\}$ and $\bigcup A_2 = \{y_1 < y_2 < \cdots < y_p\}$ where $x_i \in A_1^g$ if and only if $y_i \in A_2^g$ for all i.*

Let us begin with the following Lemma:

Lemma 4.1.14. *If $Y_{A_1}^k \cong Y_{A_2}^k$ as representations then $|\bigcup A_1| = |\bigcup A_2|$.*

Proof. Let us assume that $|\bigcup A_1| \neq |\bigcup A_2|$ and suppose without loss of generality that $n_1 = |\bigcup A_1| < |\bigcup A_2| = n_2$, then let $d = ([n_1],[n_1])$ with c_1 the trivial coloring of d, then d zeroes $Y^k_{A_2}$: given $y^T_{A_2}$, then if $T \subseteq \mathrm{dom}(d)$, $n_2 = |T| \leq |\mathrm{dom}(d)| = n_1 < n_2$, a contradiction. However, d does not zero all of $Y^k_{A_1}$, e.g. $y^{[n_1]}_{A_1}$. Therefore, the two representations are not isomorphic since no isomorphism will preserve the action of d. □

Proof of Proposition 4.1.13. Suppose that $|A^g_1| = |A^g_2|$ for all $g \in G$, $\bigcup A_1 = \{x_1 < x_2 < \cdots < x_p\}$ and $\bigcup A_2 = \{y_1 < y_2 < \cdots < y_p\}$ where $x_i \in A^g_1$ if and only if $y_i \in A^g_2$. Then we have the following map:

$$\begin{aligned}\phi : Y^k_{A_1} &\to Y^k_{A_2}\\ y^T_{A_1} &\mapsto y^T_{A_2}\end{aligned}$$

which is linearly extended to all of $Y^k_{A_1}$. Let $d' : \bigcup A_1 \to T$ and $d'' : \bigcup A_2 \to T$. Then for any $d_c \in PR_k(n;G)$,

$$d_c\,\phi(y^T_{A_1}) = d_c\,y^T_{A_2} = \begin{cases}\alpha(d''(A_2),c)\,n^\sigma y^{d(T)}_{A_2} & \text{if } T \subseteq \mathrm{dom}(d)\\ 0 & \text{if } T \nsubseteq \mathrm{dom}(d)\end{cases}$$

where $\sigma = k - \mathrm{rk}(d)$. Since the sets in A_1 and A_2 are in the same order relative to each other by hypothesis, $d''(A^g_2) = d'(A^g_1)$ for each $g \in G$, so this is

$$= \begin{cases}\alpha(d'(A_1),c)\,n^\sigma \phi(y^{d(T)}_{A_1}) & \text{if } T \subseteq \mathrm{dom}(d)\\ 0 & \text{if } T \nsubseteq \mathrm{dom}(d)\end{cases} = \begin{cases}\phi(\alpha(d'(A_1),c)\,n^\sigma y^{d(T)}_{A_1}) & \text{if } T \subset \mathrm{dom}(d)\\ \phi(0) & \text{if } T \nsubseteq \mathrm{dom}(d)\end{cases}$$

$$= \phi(d_c\,y^T_{A_1})$$

Since the $y^T_{A_1}$ and $y^T_{A_2}$ form bases of $Y^k_{A_1}$ and $Y^k_{A_2}$, respectively, this must be an isomorphism.

Now suppose that $Y^k_{A_1} \cong Y^k_{A_2}$ as representations, then Lemma 4.1.14 tells us that $|\bigcup A_1| = |\bigcup A_2|$. By hypothesis, there exists

$$\phi : Y^k_{A_1} \to Y^k_{A_2}$$

an isomorphism of representations. For each T, let

$$\phi(y^T_{A_1}) = \sum_{|U|=|T|} \lambda^T_U\, y^U_{A_2}$$

for some $\lambda^T_U \in \mathbb{C}$. Fix a T and construct the diagrams $d'_{c'}, d''_{c''} \in PR_k(G)$ from $y^T_{A_1}$ and $y^T_{A_2}$ where c' and c'' are trivial colorings of the diagrams

$$\begin{aligned}d' &: \bigcup A_1 \to T\\ d'' &: \bigcup A_2 \to T\end{aligned}$$

Also, let $e^t : [k]\backslash t \to [k]\backslash t$ and c^t be the trivial coloring of e^t.

If $T = \varnothing$ then $\bigcup A_1 = \bigcup A_2 = \varnothing$ and we are done since $Y^k_{A_\varnothing} = Y^k_{A_\varnothing}$. Otherwise, let $t \in T$. Then

$$\begin{aligned} 0 = \phi(0) &= \phi(e^t_{c^t}\, y^T_{A_1}) = e^t_{c^t}\, \phi(y^T_{A_1}) \\ &= \sum_{|U|=|T|} \lambda^T_U\, e^t_{c^t}\, y^U_{A_2} = \sum_{\substack{|U|=|T| \\ t\notin U}} \lambda^T_U\, y^U_{A_2} \end{aligned}$$

Since the $y^U_{A_2}$ are linearly independent, $\lambda^T_U = 0$ for all U such that $t \notin U$. Since $|U| = |T|$ for each U, $\lambda^T_U = 0$ for all U except $U = T$. Otherwise, $T \nsubseteq U$ which is a contradiction. So

$$\phi(y^T_{A_1}) = \lambda^T_T\, y^T_{A_2}.$$

Let $\lambda_T := \lambda^T_T$, let $T = \{t_1, t_2, \ldots, t_{|T|}\}$, and let $e^T \in PR_k$ such that $e^T : T \to T$. Let $\mathbf{1} = (1, 1, \ldots, 1) \in G$, and for each $1 \le i \le |T|$ let c_i be the coloring of e^T such that for all $1 \le \ell \le |T|$,

$$c_i(t_\ell) = \delta_{i,\ell}\mathbf{1}$$

Let $\bigcup A_1 = \{x_1 < x_2 < \cdots < x_p\}$ and $\bigcup A_2 = \{y_1 < y_2 < \cdots < y_p\}$. Suppose that there exists i such that $x_i \in A_1^{\widehat{g}}$ and $y_i \in A_2^{\widehat{h}}$ for some $\widehat{g} \neq \widehat{h}$ in G. Then $t_i = d'(x_i) \in d'(A_1^{\widehat{g}})$ and $t_i = d''(y_i) \in d''(A_2^{\widehat{h}})$, so let's hit $y^T_{A_1}$ and its image under ϕ with $e^T_{c_i}$ and see what happens:

$$\phi(e^T_{c_i}\, y^T_{A_1}) = \phi(n^{k-|T|}\, \alpha(d'(A_1), c_i)\, y^T_{A_1}) = n^{k-|T|}\, \alpha(d'(A_1), c_i)\, \lambda_T\, y^T_{A_2} \tag{4.5}$$

Since ϕ is an isomorphism of representations, this must be equal to

$$e^T_{c_i}\, \phi(y^T_{A_1}) = e^T_{c_i}(\lambda_T\, y^T_{A_2}) = n^{k-|T|}\, \alpha(d''(A_2), c_i)\lambda_T\, y^T_{A_2}, \tag{4.6}$$

but

$$\begin{aligned} \alpha(d'(A_1), c_i) &= \prod_{g\in G} \prod_{\ell\in d'(A_1^g)} \prod_{1\le j\le m} (\zeta_{q_j})^{g_j\, c_i(\ell)_j} \\ \alpha(d''(A_2), c_i) &= \prod_{g\in G} \prod_{\ell\in d''(A_2^g)} \prod_{1\le j\le m} (\zeta_{q_j})^{g_j\, c_i(\ell)_j} \end{aligned}$$

and since $c_i(\ell)$ is nonzero if and only if $\ell = t_i$,

$$\begin{aligned} \alpha(d'(A_1), c_i) &= \prod_{1\le j\le m} (\zeta_{q_j})^{\widehat{g}_j\, c_i(t_i)_j} = \prod_{1\le j\le m} (\zeta_{q_j})^{\widehat{g}_j} \\ \alpha(d''(A_2), c_i) &= \prod_{1\le j\le m} (\zeta_{q_j})^{\widehat{g}_j\, c_i(t_i)_j} = \prod_{1\le j\le m} (\zeta_{q_j})^{\widehat{h}_j} \end{aligned}$$

which must be distinct given that $\widehat{g}$ and $\widehat{h}$ are distinct. We know that (4.5) and (4.6) must be equal but the coefficients $\alpha(d'(A_1), c_i)$ and $\alpha(d''(A_2), c_i)$ are distinct. Therefore, $\lambda_T = 0$ for every T, so $\phi = 0$, which is a contradiction. Therefore, our hypothesis is false and $x_i \in A_1^g$ if and only if $y_i \in A_2^g$ for each i. □

4.1.2 Branching Rules and Bratteli Diagram

Now that we have analyzed the irreducible representations of the algebras $PR_k(n;G)$ for each k, we want to determine how they relate to each other. We can think of the monoids $PR_k(G)$ as a series of monoids all contained in each other:

$$PR_0(G) \subseteq PR_1(G) \subseteq PR_2(G) \subseteq \cdots$$

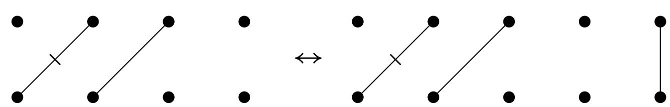
and we can think of the algebras $PR_k(n;G)$ as a series of algebras all contained in each other:

$$PR_0(n;G) \subseteq PR_1(n;G) \subseteq PR_2(n;G) \subseteq \cdots$$

where we identify the colored diagram $d_c \in PR_k(G)$ with the element $d^*_{c^*} \in PR_k(G)$ such that

$$\begin{aligned} \mathrm{dom}(d^*) &= \mathrm{dom}(d) \cup \{k\}, \\ \mathrm{img}(d^*) &= \mathrm{img}(d) \cup \{k\}, \\ c^*|_{\mathrm{img}(d)} &= c, \\ c^*(k) &= 0. \end{aligned}$$

This amounts to taking the diagram d_c, adding a vertex to the bottom and top rows on the right, and connecting these vertices with an edge colored with 0. For our Type B examples, this looks like



Let us look at the action of $PR_{k-1}(n;G)$ on Y^k_A. Let

$$\begin{aligned} X &:= \mathrm{span}\{y^T_A : k \in T\} \\ Z &:= \mathrm{span}\{y^T_A : k \notin T\} \end{aligned}$$

Both of these spaces are $PR_{k-1}(n;G)$-invariant by Proposition 4.1.4. Let us now show that both are irreducible $PR_{k-1}(n;G)$ representations.

Let $y^T_A, y^{T'}_A \in X$, then if $d : T \to T'$ and c is the trivial coloring of d, $d_c\, y^T_A = y^{T'}_A$ and $d \in PR^B_{k-1}$ since $k \in T \cap T'$, so each y^T_A generates X.

Let

$$x = \sum_{\substack{T \subseteq [k] \\ k \in T}} \lambda_T\, y^T_A \neq 0$$

with $\lambda_T \in \mathbb{C}$. Then there is some $\lambda_T \neq 0$, so let $d : T \to T$ and c be the trivial coloring of d. Then

$$d_c x = \lambda_T\, y^T_A$$

and $d_c \in PR^B_{k-1}$, so each element of X generates X so X is irreducible.

Let $y_A^T, y_A^{T'} \in Z$ and now let $d : (T \cup \{k\}) \to (T' \cup \{k\})$ and c be the trivial coloring, then $d \in PR_{k-1}^B$ and $d_c\, y_A^{T'}$ so the basis elements generate Z.

Let

$$z = \sum_{\substack{T \subseteq [k] \\ k \notin T}} \lambda_T\, y_A^T \neq 0$$

then there is a T such that λ_T so let $d : (T \cup \{k\}) \to (T \cup \{k\})$ and c the trivial coloring, then

$$d_c\, z = \lambda_T\, y_A^T$$

and $d \in PR_{k-1}^B$ so Z is also irreducible.

We can see from Proposition 4.1.13 that a set of distinct irreducible representations of $PR_k(n;G)$ is the ones corresponding to$G$-partitions A such that $\bigcup A = \varnothing$ or $\bigcup A = [\ell]$ for some $1 \leq \ell \leq k$. Let us see how these representations restrict as $PR_{k-1}(n;G)$-modules (representations).

Proposition 4.1.15. *Let $k > 0$. If A is a G-partition of $[k]$ such that $\bigcup A = [\ell]$ where $1 \leq \ell < k$, then as a $PR_{k-1}(n;G)$-module Y_A^k decomposes as:*

$$Y_A^k \cong Y_{A\setminus\{\ell\}}^{k-1} \oplus Y_A^{k-1}$$

where if $\ell \in A^h$, then $A\setminus\{\ell\}$ is the G-partition A_1 of $[k]$ with $A_1^g = A^g$ for all $g \neq h$ and $A_1^h = A^h\setminus\{\ell\}$. If $\bigcup A = \varnothing$, the left summand is dropped, and if $\ell = k$, the right summand is dropped.

Proof. The case $\bigcup A = \varnothing$ is trivial.

Let $\ell > 0$. We saw that Y_A^k breaks up into the direct sum $X \oplus Z$, as defined above. Then $X \cong Y_{A\setminus\{\ell\}}^{k-1}$ and $Z \cong Y_A^{k-1}$. If $\ell = k$ then $Z = 0$ and $X \cong Y_{A\setminus\{\ell\}}^{k-1}$. □

We see that we can index the irreducible representations Y_A^k with a sequence of elements in G:

Let the sequence $(g_1, g_2, \ldots, g_\ell)$ with $0 \leq \ell \leq k$ denote the irreducible representation Y_A^k with $i \in A^g$ iff $g_i = g$. For example, if $G = \mathbb{Z}_2$, $k = 6$ and $|\bigcup A| = \ell = 5$ with $A^1 = \{1, 3\}$ and $A^0 = \{2, 4, 5\}$, then we can represent the representation with the sequence $S = 1, 0, 1, 0, 0$ of length ℓ where we put 1s in the 1^{st} and 3^{rd} places and 0s elsewhere. Let us draw the irreducible representations of each $PR_k(n;G)$ as a graph, with these sequences representing the representations. We list the sequences representing the irreducibles of $PR_k(n;G)$ on the k^{th} row and draw an edge between the sequence S on the k^{th} row and S' on the $(k-1)^{th}$ row if the irreducible represented by S' shows up in the decomposition of the irreducible represented by S as a $PR_{k-1}(n;G)$-module. By Proposition 4.1.15, we draw an edge between S and S' if $S = S'$ or S' is the subsequence of S after removing the last element. See Figure 4.1 for the first four levels of the Bratteli diagram for $PR_k(n;\mathbb{Z}_2)$.

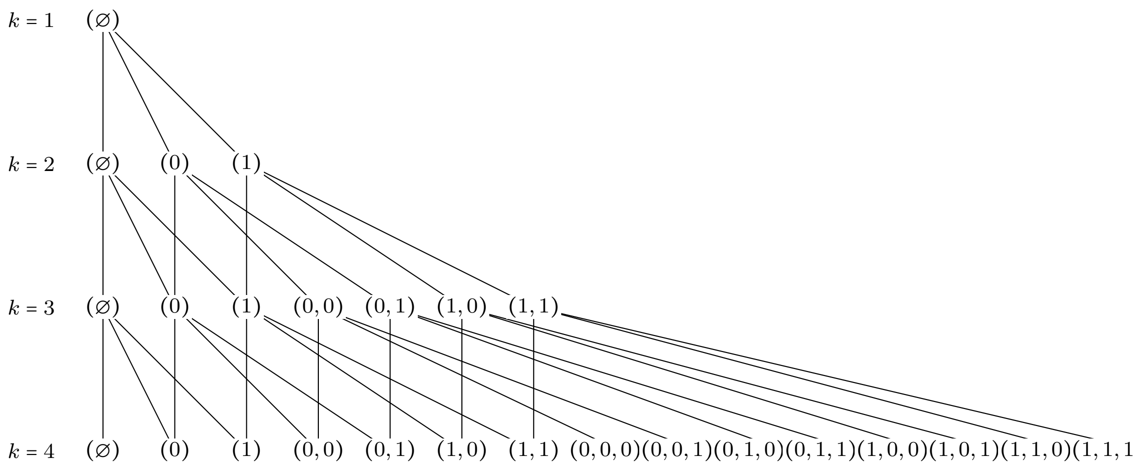


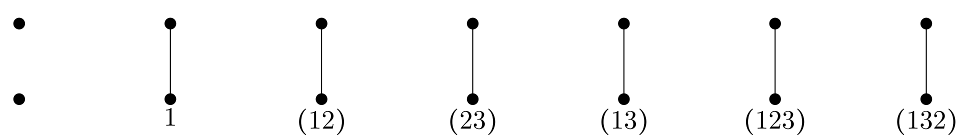
Figure 4.1: Bratteli Diagram for $G = \mathbb{Z}_2$ (Type B)

4.2 $PR_k(n; G)$ for G finite, non-abelian

We have shown in the last section that $PR_k(n; G)$ is semisimple when G is a finite abelian group. We show in this section that there exists a k and non-abelian group G such that $PR_k(n; G)$ is not semisimple for any $n \neq 0$. Note that the basis vectors y_A^T which give a complete decomposition of the regular representation into a direct sum of irreducible subrepresentations were defined in terms of the roots of unity, since every cyclic group can be embedded into the multiplicative group of complex numbers by mapping elements in the cyclic group to roots of unity. However, we cannot extend this construction to non-abelian groups.

Proposition 4.2.1. *Let S_m be the symmetric group on the set $[m]$. Then the algebra $PR_1(n; S_3)$ is not semisimple for any $n \neq 0$.*

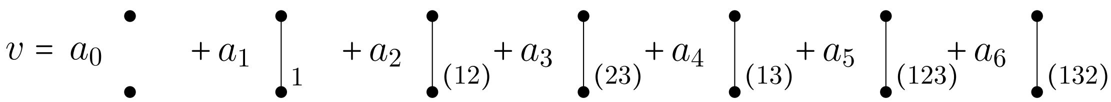
Proof. The symmetric group S_3 contains the elements 1, (12), (23), (13), (123) and (132) in cycle notation. For more information on the symmetric group and cycle notation, see [10]. Therefore, the seven colored diagrams in the algebra are



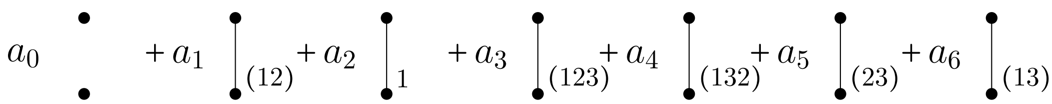
where we write the label of the edge in the diagram next to that edge. By Proposition 2.1.2, the sum of the squares of the dimensions of the irreducible representations of $PR_1(n; G)$ must add to the dimension of $PR_1(n; S_3)$. Since the set of colored diagrams form a basis of the algebra, its dimension is 7. The only way that 7 decomposes as a sum of square integers is

$$
\begin{aligned}
7 &= 2^2 + 1^2 + 1^2 + 1^2 \\
&= 1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2
\end{aligned}
$$

so if $PR_1(n;G)$ is semisimple then its regular representation is completely reducible and must either decompose into a direct sum of seven 1-dimensional representations or two 2-dimensional irreducible representations and four 1-dimensional representations. Both cases require that the regular representation of $PR_1(n;S_3)$ have at least three distinct 1-dimensional subrepresentations. We will show that there are only two distinct 1-dimensional subrepresenations of the regular representation in order to get a contradiction. Suppose that the non-zero element



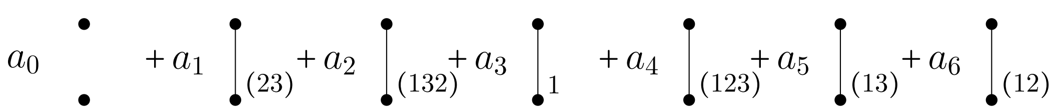
in the algebra generates a 1-dimensional representation, then if we hit v with the diagram with one edge labeled with (12), the result must be a multiple of v. After multiplying on the left with this diagram, the labels of the edges are multiplied by (12) on the left, resulting in the element



Since this is a multiple of v, suppose it is cv for some $c \in \mathbb{C}$, then

$$
\begin{aligned}
a_0 &= ca_0 \\
a_2 &= ca_1 \\
a_1 &= ca_2 \\
a_3 &= ca_5 \\
a_5 &= ca_3 \\
a_4 &= ca_6 \\
a_6 &= ca_4
\end{aligned}
$$

so $a_1 = ca_2 = c^2a_1$, $a_2 = ca_1 = c^2a_2$, and so on. Therefore, either all $a_i = 0$ for $i > 0$ for $c = 1$. Now if we hit v with the diagram with one edge colored with (23), the result is



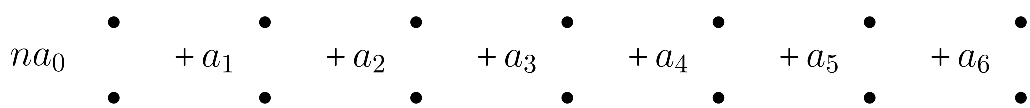
so since this is also a multiple of v, suppose it equals ev for some $e \in \mathbb{C}$, then

$$\begin{aligned} a_0 &= ea_0 \\ a_1 &= ea_3 \\ a_3 &= ea_1 \\ a_2 &= ea_6 \\ a_6 &= ea_2 \\ a_4 &= ea_5 \\ a_5 &= ea_4 \end{aligned}$$

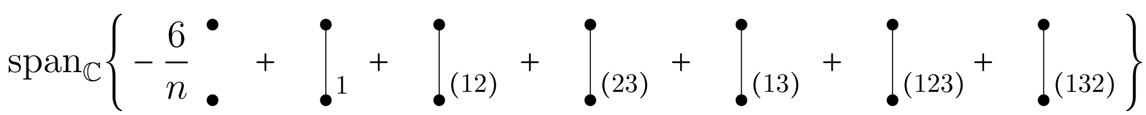
and again, either $a_i = 0$ for all $i > 0$ or $e = 1$.

If $a_i = 0$ for all $i > 0$, then since v is nonzero, $a_0 \neq 0$, so the representation generated is the set of complex multiples of the empty diagram.

If one of the a_i is nonzero, then $c = e = 1$ and by the equations above, $a_1 = a_2 = a_3 = a_4 = a_5 = a_6$. Then if we hit v with the empty diagram we get



which equals $7a_1 + na_0$ times the empty diagram. Since this must be a multiple of v and $a_1 \neq 0$ this element must equal zero, so $6a_1 + na_0 = 0$. Therefore, $a_0 = -6a_1/n$ and the representation generated by v must be equal to



Therefore, there are only two distinct 1-dimensional subrepresentations of $PR_1(n; S_3)$, a contradiction, so $PR_1(n; S_3)$ is not semisimple. □

It is unclear whether $PR_k(n; G)$ is not semisimple for any finite nonabelian group, $k > 0$ and $n \neq 0$, although we have only explored this one example in order to show that the basis of y_A^T elements constructed in this thesis which decompose $PR_k(n; G)$ for G finite abelian cannot be extended easily to all groups.

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