Difference Equations

to

Differential Equations

Numerical Approximations
of Definite Integrals

Computing a definite integral of a function f over an interval $[a, b]$ using upper and lower sums, or even as the limit of Riemann sums, is, for all but the simplest cases, a difficult task. As a result, definite integrals are almost never computed in that manner. For the most part, definite integrals are evaluated either using the Fundamental Theorem of Calculus or using numerical approximation techniques. We will take up the Fundamental Theorem of Calculus approach in the next section; in this section we consider several methods for numerical approximation.

The left-hand and right-hand rules

Recall that for an integrable function f on an interval $[a, b]$, the left-hand rule approximation for $\int_a^b f(x)dx$, using n intervals, is given by

$$A_L = h \sum_{i=0}^{n-1} f(a + ih) \tag{4.2.1}$$

and the right-hand rule approximation by

$$A_R = h \sum_{i=1}^{n} f(a + ih), \tag{4.2.2}$$

where

$$h = \frac{b - a}{n}.$$

We now look at the accuracy of these approximations. Let $x_i = a + ih$, $i = 0, 1, 2, \ldots, n$, the endpoints for a partition of $[a, b]$ using n intervals of equal length h. Assume f is continuous on $[a, b]$ and differentiable on (a, b) and that x is a point in the ith interval, that is, $x_{i-1} \leq x \leq x_i$. Then the Mean Value Theorem tells us that there exists a point c_i in the interval (x_{i-1}, x_i) such that

$$f'(c_i) = \frac{f(x) - f(x_{i-1})}{x - x_{i-1}}. \tag{4.2.3}$$

Solving for $f(x)$ in (4.2.3), we have

$$f(x) = f(x_{i-1}) + f'(c_i)(x - x_{i-1}). \tag{4.2.4}$$

Integrating both sides of (4.2.4) over the interval $[x_{i-1}, x_i]$ gives us

$$\int_{x_{i-1}}^{x_i} f(x)dx = \int_{x_{i-1}}^{x_i} f(x_{i-1})dx + \int_{x_{i-1}}^{x_i} f'(c_i)(x - x_{i-1})dx$$

$$= f(x_{i-1})(x_i - x_{i-1}) + \int_{x_{i-1}}^{x_i} f'(c_i)(x - x_{i-1})dx \qquad (4.2.5)$$

$$= f(x_{i-1})h + \int_{x_{i-1}}^{x_i} f'(c_i)(x - x_{i-1})dx,$$

where we have used the fact that the integral of a constant equals the constant multiplied by the length of the interval. Hence

$$\int_a^b f(x)dx = \sum_{i=1}^n \int_{x_{i-1}}^{x_i} f(x)dx$$

$$= \sum_{i=1}^n f(x_{i-1})h + \sum_{i=1}^n \int_{x_{i-1}}^{x_i} f'(c_i)(x - x_{i-1})dx \qquad (4.2.6)$$

$$= A_L + \sum_{i=1}^n \int_{x_{i-1}}^{x_i} f'(c_i)(x - x_{i-1})dx.$$

Thus we have

$$\left| \int_a^b f(x)dx - A_L \right| = \left| \sum_{i=1}^n \int_{x_{i-1}}^{x_i} f'(c_i)(x - x_{i-1})dx \right|$$

$$\leq \sum_{i=1}^n \left| \int_{x_{i-1}}^{x_i} f'(c_i)(x - x_{i-1})dx \right|. \qquad (4.2.7)$$
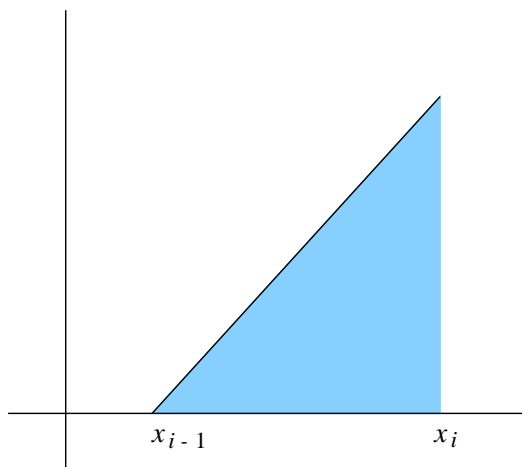
Now

$$\left| \int_{x_{i-1}}^{x_i} f'(c_i)(x - x_{i-1})dx \right| \leq \int_{x_{i-1}}^{x_i} |f'(c_i)(x - x_{i-1})|dx, \qquad (4.2.8)$$

(see Problem 11 in Section 4.1), so

$$\left| \int_a^b f(x)dx - A_L \right| \leq \sum_{i=1}^n \int_{x_{i-1}}^{x_i} |f'(c_i)(x - x_{i-1})|dx$$

$$= \sum_{i=1}^n \int_{x_{i-1}}^{x_i} |f'(c_i)|(x - x_{i-1})dx, \qquad (4.2.9)$$

where the last equality follows from the fact that $x - x_{i-1} \geq 0$ for all x in $[x_{i-1}, x_i]$. Now suppose f' is defined and continuous on $[a, b]$ and let M be the maximum value of $|f'(x)|$ for x in $[a, b]$. Then

$$\int_{x_{i-1}}^{x_i} |f'(c_i)|(x - x_{i-1})dx \leq \int_{x_{i-1}}^{x_i} M(x - x_{i-1})dx$$

$$= M \int_{x_{i-1}}^{x_i} (x - x_{i-1})dx = \frac{Mh^2}{2}, \qquad (4.2.10)$$

Figure 4.2.1 Graph of $y = x - x_{i-1}$ over the interval $[x_{i-1}, x_i]$

since the region beneath the graph of $y = x - x_{i-1}$ over the interval $[x_{i-1}, x_i]$ is a triangle with base and height both of length $h = x_i - x_{i-1}$ (see Figure 4.2.1). Substituting (4.2.10) into (4.2.9), and recalling that $h = \frac{b-a}{n}$, we have

$$\left| \int_a^b f(x)dx - A_L \right| \leq \sum_{i=1}^n \frac{Mh^2}{2} = \frac{nMh^2}{2} = \frac{nM(b-a)^2}{2n^2} = \frac{M(b-a)^2}{2n}. \tag{4.2.11}$$

In other words, the absolute value of the error of the left-hand rule approximation is bounded by a constant multiplied by $\frac{1}{n}$. This explains the behavior of the example in Section 4.1 where we saw that doubling the number of intervals would decrease the error by a factor of $\frac{1}{2}$. The same techniques yield a similar result for the right-hand rule.

The trapezoidal rule

For a decreasing function, the left-hand rule is an upper sum and the right-hand rule is a lower sum; for an increasing function, the left-hand rule is a lower sum and the right-hand rule is an upper sum. Hence, for such functions, it would seem that the average of the left-hand and right-hand rules, that is,

$$\frac{A_L + A_R}{2},$$

should provide a better approximation to $\int_a^b f(x)dx$ than either A_L or A_R. We will now show that this is true in general.

Suppose f, f', and f'' are all defined and continuous on $[a, b]$. From (4.2.4) we know that for any x in the interval $[x_{i-1}, x_i]$ there exists a point c_i in (x_{i-1}, x_i) such that

$$f(x) = f(x_{i-1}) + f'(c_i)(x - x_{i-1}). \tag{4.2.12}$$

Similarly, there exists a point d_i in (x_{i-1}, x_i) such that

$$f(x) = f(x_i) + f'(d_i)(x - x_i). \tag{4.2.13}$$

Using f' in place of f in (4.2.4), there exists a point p_i in (x_{i-1}, c_i) such that

$$f'(c_i) = f'(x_{i-1}) + f''(p_i)(c_i - x_{i-1}) \tag{4.2.14}$$

and a point q_i such that

$$f'(d_i) = f'(x_{i-1}) + f''(q_i)(d_i - x_{i-1}) \tag{4.2.15}$$

Substituting (4.2.14) into (4.2.12) and (4.2.15) into (4.2.13), we have

$$
\begin{aligned}
f(x) &= f(x_{i-1}) + (f'(x_{i-1}) + f''(p_i)(c_i - x_{i-1}))(x - x_{i-1}) \\
&= f(x_{i-1}) + f'(x_{i-1})(x - x_{i-1}) + f''(p_i)(c_i - x_{i-1})(x - x_{i-1})
\end{aligned} \tag{4.2.16}
$$

and

$$
\begin{aligned}
f(x) &= f(x_i) + (f'(x_{i-1}) + f''(q_i)(d_i - x_{i-1}))(x - x_i) \\
&= f(x_i) + f'(x_{i-1})(x - x_i) + f''(q_i)(d_i - x_{i-1})(x - x_i).
\end{aligned} \tag{4.2.17}
$$

Taking the average of (4.2.16) and (4.2.17) gives us

$$
\begin{aligned}
f(x) &= \frac{f(x) + f(x)}{2} \\
&= \frac{f(x_{i-1}) + f(x_i)}{2} + \frac{f'(x_{i-1})((x - x_{i-1}) + (x - x_i))}{2} \\
&\quad + \frac{f''(p_i)(c_i - x_{i-1})(x - x_{i-1}) + f''(q_i)(d_i - x_{i-1})(x - x_i)}{2}.
\end{aligned} \tag{4.2.18}
$$

Now

$$\int_{x_{i-1}}^{x_i} \frac{f(x_{i-1}) + f(x_i)}{2}\, dx = \frac{f(x_{i-1}) + f(x_i)}{2}(x_i - x_{i-1}) = \frac{f(x_{i-1}) + f(x_i)}{2} h \tag{4.2.19}$$

and

$$
\begin{aligned}
\int_{x_{i-1}}^{x_i} \frac{f'(x_{i-1})((x - x_{i-1}) + (x - x_i))}{2}\, dx &= f'(x_{i-1}) \int_{x_{i-1}}^{x_i} \left(x - \frac{x_{i-1} + x_i}{2} \right) dx \\
&= 0,
\end{aligned} \tag{4.2.20}
$$

where the final equality follows from the fact that region between the graph of
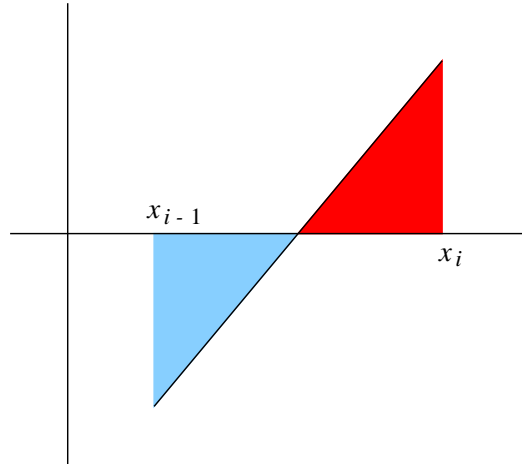
$$y = x - \frac{x_{i-1} + x_i}{2}$$

Figure 4.2.2 Graph of $y = x - \dfrac{x_{i-1} + x_i}{2}$ over the interval $[x_{i-1}, x_i]$

and the interval $[x_{i-1}, x_i]$ forms two triangles of equal area, one above the x-axis and one below (see Figure 4.2.2). Moreover, if K is the maximum value of $|f''(x)|$ for x in $[a, b]$, then

$$\left| \frac{f''(p_i)(c_i - x_{i-1})(x - x_{i-1}) + f''(q_i)(d_i - x_{i-1})(x - x_i)}{2} \right| \leq \frac{K}{2}(|c_i - x_{i-1}||x - x_{i-1}|$$
$$+ |d_i - x_{i-1}||x - x_i|).$$

Since the points c_i, d_i, and x are all in $[x_{i-1}, x_i]$, it follows that

$$\left| \frac{f''(p_i)(c_i - x_{i-1})(x - x_{i-1}) + f''(q_i)(d_i - x_{i-1})(x - x_i)}{2} \right| \leq Kh^2. \qquad (4.2.21)$$

Hence

$$\int_{x_{i-1}}^{x_i} \left| \frac{f''(p_i)(c_i - x_{i-1})(x - x_{i-1}) + f''(q_i)(d_i - x_{i-1})(x - x_i)}{2} \right| dx \leq \int_{x_{i-1}}^{x_i} Kh^2 dx.$$

Now

$$\int_{x_{i-1}}^{x_i} Kh^2 dx = Kh^2(x_i - x_{i-1}) = Kh^3, \qquad (4.2.22)$$

so we have

$$\int_{x_{i-1}}^{x_i} \left| \frac{f''(p_i)(c_i - x_{i-1})(x - x_{i-1}) + f''(q_i)(d_i - x_{i-1})(x - x_i)}{2} \right| dx \leq Kh^3. \qquad (4.2.23)$$

Putting this all together, we have

$$
\begin{aligned}
\int_a^b f(x)dx &= \sum_{i=1}^n \int_{x_{i-1}}^{x_i} f(x)dx \\
&= \sum_{i=1}^n \int_{x_{i-1}}^{x_i} \frac{f(x_{i-1}) + f(x_i)}{2}\, dx \\
&\quad + \sum_{i=1}^n \int_{x_{i-1}}^{x_i} \frac{f'(x_{i-1})((x - x_{i-1}) + (x - x_i))}{2}\, dx \\
&\quad + \sum_{i=1}^n \int_{x_{i-1}}^{x_i} \frac{f''(p_i)(c_i - x_{i-1})(x - x_{i-1}) + f''(q_i)(d_i - x_{i-1})(x - x_i)}{2}\, dx \\
&= \sum_{i=1}^n \frac{f(x_{i-1}) + f(x_i)}{2} h + 0 \\
&\quad + \sum_{i=1}^n \int_{x_{i-1}}^{x_i} \frac{f''(p_i)(c_i - x_{i-1})(x - x_{i-1}) + f''(q_i)(d_i - x_{i-1})(x - x_i)}{2}\, dx \\
&= \frac{h \sum_{i=1}^n f(x_{i-1}) + h \sum_{i=1}^n f(x_i)}{2} \\
&\quad + \sum_{i=1}^n \int_{x_{i-1}}^{x_i} \frac{f''(p_i)(c_i - x_{i-1})(x - x_{i-1}) + f''(q_i)(d_i - x_{i-1})(x - x_i)}{2}\, dx \\
&= \frac{A_L + A_R}{2} \\
&\quad + \sum_{i=1}^n \int_{x_{i-1}}^{x_i} \frac{f''(p_i)(c_i - x_{i-1})(x - x_{i-1}) + f''(q_i)(d_i - x_{i-1})(x - x_i)}{2}\, dx,
\end{aligned}
$$

from which it follows, using (4.2.23), that

$$
\left| \int_a^b f(x)dx - \frac{A_L + A_R}{2} \right| \leq \sum_{i=1}^n Kh^3 = nKh^3 = nK \left(\frac{b-a}{n} \right)^3 = \frac{K(b-a)^3}{n^2}. \tag{4.2.24}
$$

That is, if we approximate $\int_a^b f(x)dx$ by

$$
\frac{A_L + A_R}{2},
$$

then the absolute value of the error is bounded by a constant multiplied by $\frac{1}{n^2}$. In particular, if we double the number of intervals, we should expect the error to decrease by a factor of

$$
\left(\frac{1}{2} \right)^2 = \frac{1}{4}.
$$

We call

$$A_T = \frac{A_L + A_R}{2} \qquad (4.2.25)$$

a *trapezoidal rule* approximation for $\int_a^b f(x)dx$. The name comes from the fact that (4.2.25) may also be derived by replacing the areas of rectangles by the areas of trapezoids in the Riemann sum approximations (see Problem 6).

Example   In Section 4.1 we saw that the left-hand and right-hand rule approximations for $\int_{-1}^{2}(x^2 + 1)dx$ using $n = 6$ intervals are $A_L = 5.375$ and $A_R = 6.875$. Hence the corresponding trapezoidal rule approximation is

$$A_T = \frac{5.375 + 6.875}{2} = 6.125.$$

With $n = 100$, the left-hand and right-hand rules give us $A_L = 5.95545$ and $A_R = 6.04545$, yielding a trapezoidal rule approximation of

$$A_T = \frac{5.95545 + 6.04545}{2} = 6.00045.$$

Example   In a Section 4.1 we approximated

$$A = \int_1^{10} \frac{1}{t}\, dt$$

using both left-hand and right-hand rules and, after noting that to 6 decimal places $A = 2.302585$, obtained the following table of values:

n	A_R	A_L	$|A - A_R|$	$|A - A_L|$
10	1.960214	2.770214	0.342371	0.467629
20	2.116477	2.521477	0.186108	0.218892
40	2.205491	2.407991	0.097094	0.105406
80	2.253003	2.354253	0.049582	0.051668
160	2.277534	2.328159	0.025052	0.025574
320	2.289994	2.315307	0.012591	0.012722

Using these results, we may compute the following trapezoidal rule approximations:

n	A_T	$|A - A_T|$
10	2.365214	0.062629
20	2.318977	0.016392
40	2.306741	0.004156
80	2.303628	0.001043
160	2.302846	0.000265
320	2.302650	0.000065

We see that the errors in the trapezoidal rule approximations are significantly smaller than the corresponding errors for the left-hand and right-hand rule approximations. Moreover,

in agreement with our work above, the errors in the trapezoidal rule approximations decrease by a factor of, roughly, $\frac{1}{4}$ when we double the number of intervals. Hence we see the trapezoidal rule approximations converging to the value of the definite integral at a significantly faster rate than do the left-hand and right-hand rule approximations.

The midpoint rule

As above, let f be an integrable function on an interval $[a, b]$, n a positive integer, $h = \frac{b-a}{n}$, and, for $i = 0, 1, 2, \ldots, n$, $x_i = a + ih$, (the endpoints of a partition of $[a, b]$ with n intervals of equal length h). We may think of the trapezoidal rule as improving on the left-hand and right-hand rules by approximating the area of the region beneath the graph of f and above the interval $[x_{i-1}, x_i]$ using a rectangle with height equal to the average of $f(x_{i-1})$ and $f(x_i)$. Another approach is to average x_{i-1} and x_i before evaluating f. Since each interval is of length h, we may find the midpoint by adding $\frac{h}{2}$ to the left-hand endpoint. Namely, if we let

$$c_1 = a + \frac{h}{2},$$

$$c_2 = x_1 + \frac{h}{2} = a + h + \frac{h}{2} = a + \frac{3}{2}h,$$

$$c_3 = x_2 + \frac{h}{2} = a + 2h + \frac{h}{2} = a + \frac{5}{2}h,$$

$$\vdots \qquad\qquad\qquad (4.2.26)$$

$$c_i = x_{i-1} + \frac{h}{2} = a + (i-1)h + \frac{h}{2} = a + \left(i - \frac{1}{2}\right)h,$$

$$\vdots$$

$$c_n = x_{n-1} + \frac{h}{2} = a + (n-1)h + \frac{h}{2} = a + \left(n - \frac{1}{2}\right)h,$$

then c_i is the midpoint of the ith interval $[x_{i-1}, x_i]$. We call

$$A_M = \sum_{i=1}^{n} f(c_i)h = h \sum_{i=1}^{n} f\left(a + \left(i - \frac{1}{2}\right)h\right), \qquad (4.2.27)$$

the Riemann sum formed by evaluating f at the points c_1, c_2, c_3, $\ldots$, c_n, a *midpoint rule approximation* of the definite integral $\int_a^b f(x)dx$.

Example   To find the midpoint rule approximation for $\int_{-1}^{2}(x^2 + 1)dx$ using $n = 6$ intervals, we would have

$$h = \frac{2 - (-1)}{6} = 0.5.$$

Then the interval endpoints are $x_0 = -1$, $x_1 = -0.5$, $x_2 = 0$, $x_3 = 0.5$, $x_4 = 1$, $x_5 = 1.5$, and $x_6 = 2$, from which we find the midpoints $c_1 = -0.75$, $c_2 = -0.25$, $c_3 = 0.25$, $c_4 = 0.75$,
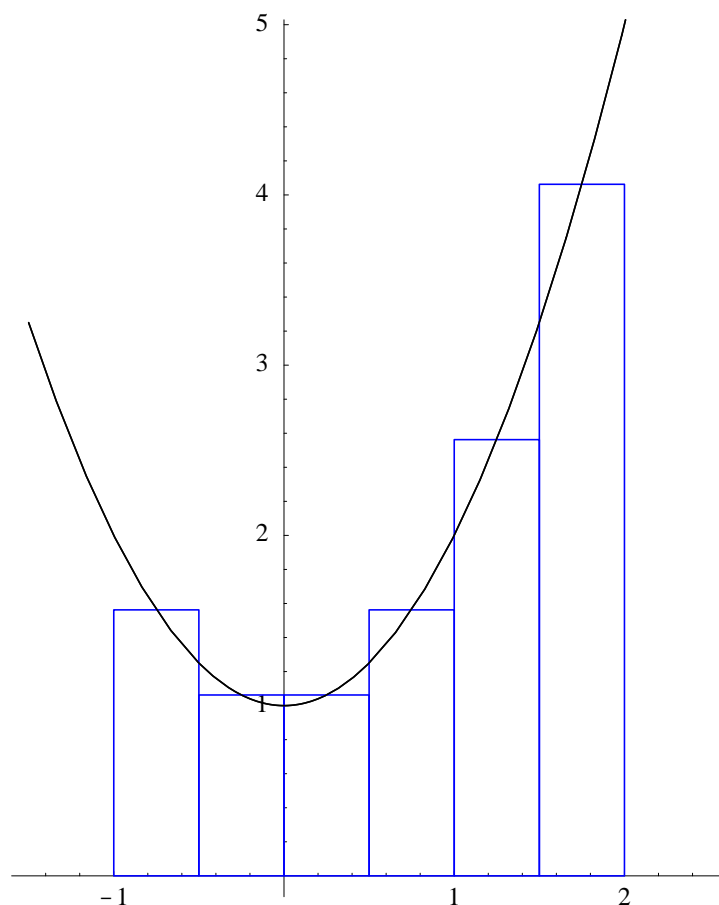
Figure 4.2.3 Midpoint rule approximation for $\displaystyle\int_{-1}^{2}(x^2+1)dx$

$c_5 = 1.25$, and $c_6 = 1.75$ (see Figure 4.2.3). Thus, letting $f(x) = x^2 + 1$, we have

$$\begin{aligned}
A_M &= 0.5(f(-0.75) + f(-0.25) + f(0.25) + f(0.75) + f(0.125) + f(1.75)) \\
&= 0.5(1.5625 + 1.0625 + 1.0625 + 1.5625 + 2.5625 + 4.0625) \\
&= 5.9375.
\end{aligned}$$

With $n = 100$ intervals, using (4.2.27) with a computer, we have $A_M = 5.999775$.

Example   Applying the midpoint rule to the problem of approximating

$$A = \int_{1}^{10} \frac{1}{t}\, dt,$$

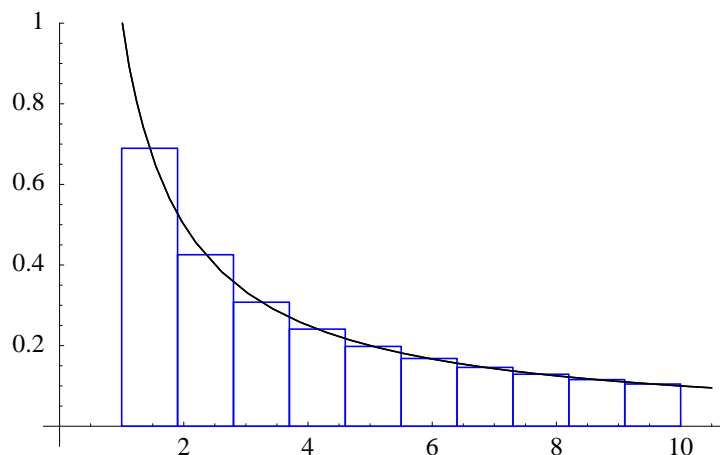we obtain the following table (again rounded to 6 decimal places):

Figure 4.2.4 Midpoint rule approximation for $\displaystyle\int_1^{10} \frac{1}{t}\, dt$

n	A_M	$|A - A_M|$
10	2.272740	0.029845
20	2.294504	0.008081
40	2.300515	0.002070
80	2.302064	0.000521
160	2.302455	0.000130
320	2.302552	0.000033

Notice that, as with the trapezoidal rule, doubling the number of intervals decreases the error by a factor of about $\frac{1}{4}$. Moreover, note that the error in each approximation is approximately $\frac{1}{2}$ of the corresponding error for the trapezoidal rule.

An analysis of the error in the midpoint rule, similar to that which we did above for the left-hand, right-hand, and trapezoidal rules, would show that the absolute value of the error is bounded by a constant multiplied by $\frac{1}{n^2}$. Hence doubling the number of intervals will decrease the error by, roughly, a factor of $\frac{1}{4}$, as was evidenced in the previous example. Moreover, a more careful examination of the error (one requiring the use of Taylor polynomials, which we will discuss in Chapter 5) would show that there is a sense in which it is typically on the order of $\frac{1}{2}$ the size of the error of the trapezoidal rule.

Simpson's rule

We saw above that averaging the left-hand and right-hand rules, two approximation methods with errors bounded by a constant multiple of $\frac{1}{n}$, resulted in an approximation method, the trapezoidal rule, with an error bounded by a constant multiple of $\frac{1}{n^2}$. We might now think that we could improve on the trapezoidal and midpoint rules, two rules with errors bounded by a constant multiple of $\frac{1}{n^2}$, by computing their average. However, it turns out that the relationship between these two rules is not as simple as with the left-hand and right-hand rules; in fact, one really needs to use Taylor polynomials in order to understand the error terms fully. At the same time, there is a hint in our previous example. Given

that the error from the midpoint rule was about $\frac{1}{2}$ of the error of the trapezoidal rule, it would be reasonable to guess that perhaps an average of the two which gives twice as much weight to the midpoint rule would be appropriate. This in fact turns out to be the right mixture, and we define

$$A_S = \frac{1}{3}A_T + \frac{2}{3}A_M = \frac{A_T + 2A_M}{3}. \tag{4.2.28}$$

We call A_S a *Simpson's rule approximation* for $\int_a^b f(x)dx$. This method of approximating definite integrals is named for Thomas Simpson (1710-1761). Simpson developed this rule in 1743 as a method for approximating the area under a curve after first approximating the curve with a number of parabolic arcs.

Example   Using $n = 6$ intervals, we saw above that the trapezoidal rule approximation for $\int_{-1}^{2}(x^2 + 1)dx$ is $A_T = 6.1250$ and the midpoint rule approximation is $A_M = 5.9375$. Thus the corresponding Simpson's rule approximation is

$$A_S = \frac{6.1250 + (2)(5.9375)}{3} = \frac{18}{3} = 6.$$

With $n = 100$ intervals, we have $A_T = 6.000450$ and $A_M = 5.999775$, giving us

$$A_S = \frac{6.000450 + (2)(5.999775)}{3} = \frac{18}{3} = 6.$$

It may seem surprising in this example that we get the same result with 100 intervals as we do with 6, but in fact this is the exact answer. It may be shown, either by careful examination of the error or by deriving the rule from parabolic approximations, that Simpson's rule will find the exact value for the definite integral of any quadratic polynomial. What is even more surprising, careful examination of the error using Taylor polynomials shows that Simpson's rule is exact for cubic polynomials as well.

Example   Using the values for the trapezoidal and midpoint rule approximations obtained above, we have the following approximations for

$$A = \int_1^{10} \frac{1}{t}\, dt$$

using Simpson's rule:

n	A_S	$|A - A_S|$
10	2.303565	0.000980
20	2.302662	0.000077
40	2.302590	0.000005
80	2.302585	0.000000

We have stopped the table at 80 intervals because at that point the approximation is accurate to 6 decimal places.
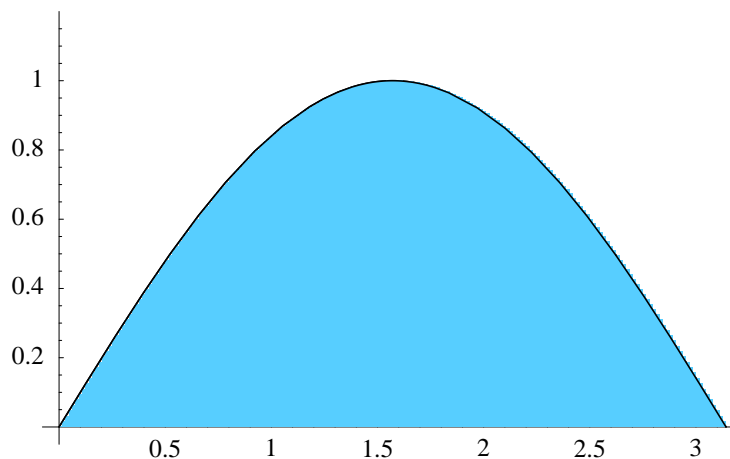
Figure 4.2.5 Region beneath $y = \sin(x)$ over the interval $[0, \pi]$

It may be shown that the absolute value of the error using Simpson's rule is bounded by a constant multiple of $\frac{1}{n^4}$, resulting in a dramatic improvement over both the trapezoidal and midpoint rules. For Simpson's rule, doubling the number of intervals typically decreases the error by a factor of

$$\left(\frac{1}{2}\right)^4 = \frac{1}{16},$$

a general fact for which we can see some evidence in the preceding example. To be fair, since Simpson's rule makes use of both the trapezoidal rule and the midpoint rule approximations, the function being integrated must be evaluated both at the endpoints and at the midpoint of each interval. This requires evaluating the function at $2n + 1$ points, whereas the trapezoidal rule evaluates the function at $n + 1$ points and the midpoint rule evaluates the function at n points. Thus, for direct comparison of errors, one should compare, for example, Simpson's rule with 10 subdivisions to the other rules using 20 subdivisions. Nevertheless, Simpson's rule converges to the value of the integral much faster than the other methods and, hence, is the method of preference among the ones we have discussed. Even faster methods exist, but we will leave them for a more advanced course.

When approximating the value of an integral, there is in general no practical way to know how many intervals are necessary in order to obtain an approximation to a desired level of accuracy. Analogous to the way in which we applied Newton's method, we normally compute a sequence of approximations, perhaps starting with only two intervals and then doubling the number of intervals from one approximation to the next, stopping when we obtain two successive approximations whose difference, in absolute value, is less than the desired level of accuracy. The next example illustrates this procedure.

Example   Suppose we wish to approximate, with an error less than 0.0001, the area A of the region between one arch of the curve $y = \sin(x)$ and the x-axis, as shown in Figure 4.2.5. That is, we want to find

$$A = \int_0^\pi \sin(x)dx.$$

Starting with $n = 2$ intervals and using Simpson's rule, we generate the following table of approximations, rounding to 6 decimal places:

n	A_S
2	2.004560
4	2.000269
8	2.000017
16	2.000001

Since the absolute value of the difference between the last two approximations is less than 0.0001, we stop at this point and use 2.0000 as our approximation for A.

Problems

1. Approximate each of the following integrals using the trapezoidal and midpoint rules with $n = 4$ intervals.

   (a) $\int_0^1 x^2 dx$ (b) $\int_0^\pi \sin(x) dx$

   (c) $\int_1^5 \frac{1}{t} dt$ (d) $\int_{-1}^1 z^3 dz$

   (e) $\int_0^4 (t^2 + t) dt$ (f) $\int_0^2 \sqrt{4 - x^2} dx$

2. Use your results from Problem 1 to compute the corresponding Simpson's rule approximation for each integral.

3. Approximate each of the following integrals using the trapezoidal and midpoint rules with $n = 50$ intervals.

   (a) $\int_0^1 x^2 dx$ (b) $\int_0^\pi \sin(x) dx$

   (c) $\int_{-2}^5 (4x^2 + 3x - 6) dx$ (d) $\int_1^2 \frac{\sin(x)}{x} dx$

   (e) $\int_0^\pi x^2 \cos(x) dx$ (f) $\int_0^{2\pi} \sqrt{1 - \sin^2(t)}\ dt$

   (g) $\int_{-5}^5 \frac{1}{x^2 + 1} dx$ (h) $\int_{-\pi}^\pi \sin(3x)\cos(x) dx$

4. Use your results from Problem 3 to compute the corresponding Simpson's rule approximation for each integral.

5. Approximate the following definite integrals using Simpson's rule. Starting with $n = 2$ intervals, compute a sequence of approximations by doubling the number of intervals from one approximation to the next. Stop when the absolute value of the difference between two successive approximations is less than 0.00001.

(a) $\displaystyle\int_1^4 x^2\,dx$

(b) $\displaystyle\int_0^6 (3x^2 + 4x - 3)\,dx$

(c) $\displaystyle\int_0^\pi \sin^2(x)\,dx$

(d) $\displaystyle\int_0^{2\pi} \sin^2(x)\cos^2(x)\,dx$

(e) $\displaystyle\int_{-\pi}^\pi \sqrt{1 + \cos^2(\theta)}\,d\theta$

(f) $\displaystyle\int_{0.1}^2 \frac{\sin(x)}{x}\,dx$

6. Suppose f is integrable on the interval $[a, b]$. Divide $[a, b]$ into n equal intervals of length $h = \frac{b-a}{n}$ and let $x_0, x_1, x_2, \ldots, x_n$ be the endpoints of these intervals. Let A_T be the trapezoidal rule approximation for $\int_a^b f(x)\,dx$.

   (a) Show that

   $$A_T = \frac{h}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)).$$

   (b) Assume $f(x) \geq 0$ for all x in $[a, b]$. For $i = 1, 2, 3, \ldots, n$, let A_i be the area of the trapezoid with vertices at $(x_{i-1}, 0)$, $(x_{i-1}, f(x_{i-1}))$, $(x_i, f(x_i))$, and $(x_i, 0)$ (that is, A_i is the area of a trapezoid with one side being the interval $[x_{i-1}, x_i]$ and parallel sides extending from the x-axis up to the graph of f). Then we could approximate $\int_a^b f(x)\,dx$ by $A_1 + A_2 + A_3 + \cdots + A_n$. Show that

   $$A_T = A_1 + A_2 + A_3 + \cdots + A_n.$$

7. Suppose f is integrable on the interval $[a, b]$. Divide $[a, b]$ into $2n$ equal intervals of length $h = \frac{b-a}{2n}$ and let $x_0,, x_1, x_2, \ldots, x_{2n}$ be the endpoints of these intervals. Let A_T and A_M be the trapezoidal rule and midpoint rule approximations for $\int_a^b f(x)\,dx$ using the n intervals with endpoints $x_0, x_2, x_4, \ldots, x_{2n}$. Let A_S be the corresponding Simpson's rule approximation. Show that

$$A_S = \frac{h}{3}(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{2n-2}) + 4f(x_{2n-1}) + f(x_{2n})).$$

8. Let $T(t)$ be the temperature at t hours after midnight at the Kalispell airport and suppose the following values for T were recorded on March 15, 1955:

Time (t)	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5
Temperature (T)	40	38	37	36	33	30	28	28	27	26

Time (t)	5.0	5.5	6.0	6.5	7.0	7.5	8.0	8.5	9.0	9.5
Temperature (T)	24	26	30	30	32	35	37	38	40	45

Time (t)	10.0	10.5	11.0	11.5	12.0
Temperature (T)	47	47	48	49	50

   (a) Approximate $\int_0^{12} T(t)\,dt$ using Simpson's rule. You may wish to use the formula in Problem 7.

(b) What does

$$A = \frac{1}{12} \int_0^{12} T(t)dt$$

represent?

(c) How does A compare with $\frac{1}{25} \sum_{t=0}^{24} T\left(\frac{t}{2}\right)$?

9. Find the area beneath one arch of the curve $y = \sin^2(x)$.

10. Let R be the region in the plane bounded by the curves $y = x^2$ and $y = (x - 2)^2$ and the x-axis. Find the area of R.