Difference Equations

to

Differential Equations

Section 3.2

Best Affine Approximations, Derivatives, and Rates of Change

In this section we will take up the general question of how to find best affine approximations and also discuss an interpretation of the derivative of a function as an instantaneous rate of change. We will consider specific computational procedures for finding derivatives in Sections 3.3 through 3.5.

To begin, suppose f is a function defined on an open interval containing the point c and let T be an affine function with $T(c) = f(c)$. As in Section 3.1, we may write T in the form

$$T(x) = m(x - c) + f(c) \tag{3.2.1}$$

for some constant m. Let

$$R(h) = f(c + h) - T(c + h) = f(c + h) - mh - f(c). \tag{3.2.2}$$

Then

$$\begin{aligned} \lim_{h\to 0} \frac{R(h)}{h} &= \lim_{h\to 0} \frac{f(c+h) - T(c+h)}{h} \\ &= \lim_{h\to 0} \frac{f(c+h) - mh - f(c)}{h} \\ &= \lim_{h\to 0} \left(\frac{f(c+h) - f(c)}{h} - m \right). \end{aligned} \tag{3.2.3}$$

Hence $R(h)$ is $o(h)$, and T is the best affine approximation to f at c, if and only if

$$\lim_{h\to 0} \left(\frac{f(c+h) - f(c)}{h} - m \right) = 0, \tag{3.2.4}$$

which is true if and only if

$$\lim_{h\to 0} \frac{f(c+h) - f(c)}{h} = m. \tag{3.2.5}$$

In particular, if

$$\lim_{h\to 0} \frac{f(c+h) - f(c)}{h}$$

exists, then f has a best affine approximation at c and

$$f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h}. \tag{3.2.6}$$

Conversely, if $T(x) = m(x-c) + f(c)$ is the best affine approximation to f at c, then it follows that

$$m = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}. \tag{3.2.7}$$

Definition We say a function f is *differentiable* at a point c if

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \tag{3.2.8}$$

exists.

In summary, if we are given a function f which is differentiable at c, then the best affine approximation to f at c exists and is given by

$$T(x) = f'(c)(x-c) + f(c), \tag{3.2.9}$$

where

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}. \tag{3.2.10}$$

Conversely, if f has a best affine approximation at a point c, then f is differentiable at c and the best affine approximation is given by (3.2.9).

Example Consider the problem of finding the best affine approximation to $f(x) = x^2$ at $x = 1$, a problem we first looked at in Section 1.1. We first need to find the derivative $f'(1)$. Using (3.2.10), we have

$$\begin{aligned} f'(1) &= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h \to 0} \frac{(1+h)^2 - 1}{h} \\ &= \lim_{h \to 0} \frac{1 + 2h + h^2 - 1}{h} \\ &= \lim_{h \to 0} \frac{h(2+h)}{h} \\ &= \lim_{h \to 0} (2+h) \\ &= 2. \end{aligned}$$

From (3.2.9), it now follows that the best affine approximation to f at 1 is

$$T(x) = 2(x-1) + 1 = 2x - 1.$$

Furthermore, from our discussion in Section 3.1, the equation of the line tangent to the graph of $y = x^2$ at the point $(1,1)$ is then

$$y = 2x - 1,$$

as shown in Figure 3.2.1.

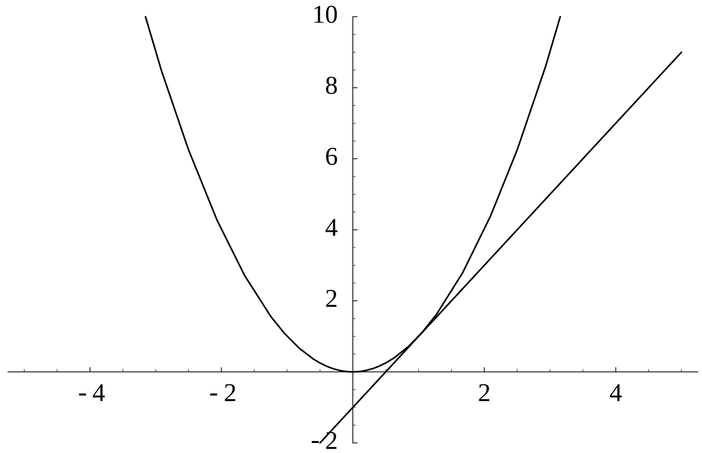

Figure 3.2.1 Graphs of $y = x^2$ and $y = 2x - 1$

Frequently we will be interested in the derivative of a function not just at a single point, but at many different points. Instead of performing the above calculation at each point separately, we try to compute the derivative at an arbitrary point, after which we can substitute in any desired point for evaluation. In fact, for any given function f, we may define a new function f' by setting

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{3.2.11}$$

for all points x at which the limit exists. This new function, f', is called the *derivative* of f. Note that the domain of f' may be smaller than the domain of f. If the open interval (a, b) is in the domain of f', we say f is *differentiable* on (a, b).

Example Let $f(x) = \sqrt{x}$. From our work in the Section 3.1 we know that

$$f'(1) = \frac{1}{2}.$$

Now we will find a general expression for $f'(x)$ at an arbitrary point x in $(0, \infty)$. Using (3.2.11), we have

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}\,\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} \\
&= \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} \\
&= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h}+\sqrt{x})} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} \\
&= \frac{1}{\sqrt{x}+\sqrt{x}} \\
&= \frac{1}{2\sqrt{x}}.
\end{aligned}$$

Hence f is differentiable on $(0,\infty)$. In particular, we once again have

$$f'(1) = \frac{1}{2}.$$

Moreover, it is now straightforward to find the best affine approximation to f at any point $c > 0$. For example,

$$f'(16) = \frac{1}{8},$$

so the best affine approximation to $f(x) = \sqrt{x}$ at $x = 16$ is

$$T(x) = \frac{1}{8}(x-16) + 4 = \frac{1}{8}x + 2.$$

See Figure 3.2.2 for the graphs of f and T.

Example Now consider $g(t) = t^3$. Then

$$\begin{aligned}
g'(t) &= \lim_{h\to 0} \frac{g(t+h)-g(t)}{h} \\
&= \lim_{h\to 0} \frac{(t+h)^3-t^3}{h} \\
&= \lim_{h\to 0} \frac{t^3+3t^2h+3th^2+h^3-t^3}{h} \\
&= \lim_{h\to 0} \frac{h(3t^2+3th+h^2)}{h} \\
&= \lim_{h\to 0} (3t^2+3th+h^2) \\
&= 3t^2.
\end{aligned}$$

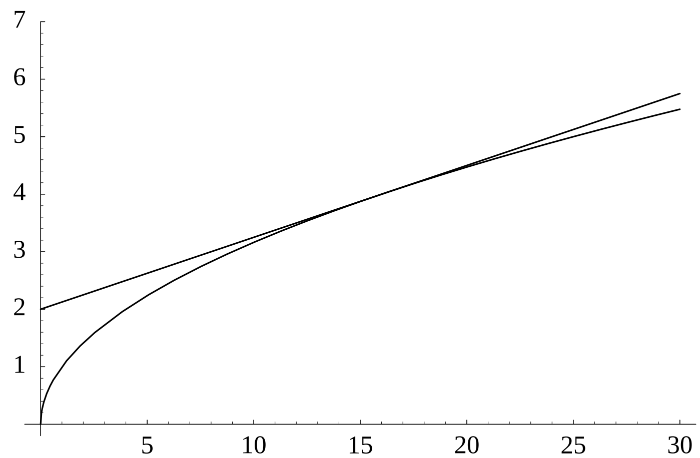

Figure 3.2.2 Graphs of $f(x) = \sqrt{x}$ and $T(x) = \frac{1}{8}x + 2$

Hence, for example, $g'(-2) = 12$, and the best affine approximation to $g(t) = t^3$ at $t = -2$ is

$$T(t) = 12(t + 2) - 8 = 12t + 16.$$

See Figure 3.2.3 for the graphs of g and T.

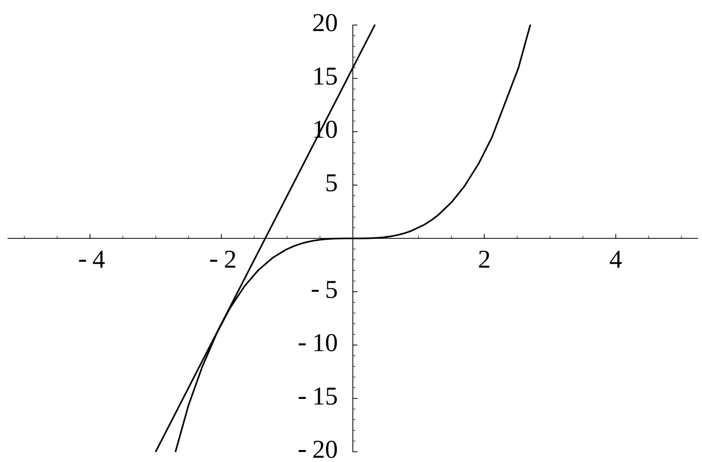

Figure 3.2.3 Graphs of $g(t) = t^3$ and $T(t) = 12t + 16$

Example Suppose we wish to find the best affine approximation to $f(x) = |x|$ at $x = 0$. To find the derivative of f at 0, we need to consider the quotient

$$\frac{f(0+h) - f(0)}{h} = \frac{|h|}{h} = \begin{cases} \dfrac{-h}{h} = -1, & \text{if } h < 0, \\ \dfrac{h}{h} = 1, & \text{if } h > 0. \end{cases}$$

Thus

$$\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = -1$$

and

$$\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = 1,$$

from which it follows that

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

does not exist. In other words, f is not differentiable at 0. Thus f does not have a best affine approximation at 0. However, for $x < 0$, the graph of f is a straight line with slope -1 and for $x > 0$, the graph of f is a straight line with slope 1. Thus

$$f'(x) = \begin{cases} -1, & \text{if } x < 0, \\ 1, & \text{if } x > 0. \end{cases}$$

Hence the domain of f' is $\{x \mid x \neq 0\}$, whereas the domain of f is the interval $(-\infty, \infty)$.

The previous example illustrates the fact that a function may be continuous at a point, as $f(x) = |x|$ is continuous at $x = 0$, without being differentiable at that point. However, it turns out that if a function is differentiable at a point, then it must be continuous at that point. To see this, note that if T is the best affine approximation to a function f at c and $r(x) = f(x) - T(x)$ is the remainder function, then

$$f(x) = T(x) + r(x). \tag{3.2.12}$$

Since T is a continuous function, $\lim\limits_{x \to c} r(x) = 0$, and $T(c) = f(c)$, we have

$$\lim_{x \to c} f(x) = \lim_{x \to c} (T(x) + r(x)) = \lim_{x \to c} T(x) + \lim_{x \to c} r(x) = T(c) + 0 = f(c), \tag{3.2.13}$$

which is what it means for f to be continuous at c.

Proposition If f is differentiable at a point c, then f is continuous at c.

Leibniz notation and rates of change

If $y = f(x)$ with $f(x) = mx + b$, then one unit change in x results in m units of change in y. That is, for a straight line, the slope of the line is the rate of change of y with respect to x. Moreover, since f is its own best affine approximation (and a straight line is its own tangent line), we have $f'(x) = m$ for all values of x. Hence, in this case, the derivative of f gives the rate of change of y with respect to x. What distinguishes this type of function from other functions, and what makes the slope easily computable, is that this rate of change is a constant. We will now use derivatives to give meaning to the rate of change of an arbitrary function at a point where it is differentiable.

If $y = f(x)$, it is common to write Δx for an increment in x and Δy for the change in y corresponding to a change in x of Δx. In our notation above, we would write

$$\Delta x = h \tag{3.2.14}$$

and

$$\Delta y = f(x + \Delta x) - f(x). \tag{3.2.15}$$

Thus we can write

$$\frac{f(x+h) - f(x)}{h} = \frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{\Delta y}{\Delta x}, \tag{3.2.16}$$

from which we have

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}. \tag{3.2.17}$$

This type of notation, although not this type of reasoning, motivated Leibniz to denote $f'(x)$ by

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}. \tag{3.2.18}$$

If the derivative is to be evaluated at a point c, then we would write

$$f'(c) = \left.\frac{dy}{dx}\right|_{x=c}. \tag{3.2.19}$$

Example If $y = \sqrt{x}$, then from our result above we may write

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

and, for example,

$$\left.\frac{dy}{dx}\right|_{x=9} = \frac{1}{2\sqrt{9}} = \frac{1}{6}.$$

Now $\frac{\Delta y}{\Delta x}$ represents the average rate of change of y over the interval from x to $x + \Delta x$. That is, this ratio tells us how much y changes per unit change in x over the interval. As we let Δx go to 0, this ratio will approach a limiting value, namely, the derivative , which we may interpret as the *instantaneous rate of change* of y with respect to x. If the rate of change of y with respect to x were not to change over an interval of length 1, then y would change by an amount equal to $\frac{dy}{dx}$ over that interval.

As an example, if $s = f(t)$ gives the position of an object moving in a straight line, then $\frac{\Delta s}{\Delta t}$ is the average rate of change of position of the object with respect to time, which we call its *average velocity*. Then $\frac{ds}{dt}$, the derivative of s with respect to t, is the instantaneous rate of change of position with respect to time; that is, $\frac{ds}{dt}$ is the *instantaneous velocity*, or, simply, *velocity*, of the object. The difference between $\frac{\Delta s}{\Delta t}$ and $\frac{ds}{dt}$ is the difference between finding the average speed for a trip in a car by dividing the total miles traveled by the total time elapsed and finding the instantaneous speed at any one time during the trip by looking at the car's speedometer.

Example Galileo discovered that if an object is dropped from a initial height of 100 feet, then, ignoring the effects of air resistance, its height, in feet, above the ground after t seconds would be

$$s = 100 - 16t^2.$$

For example, at time $t = 1$ the object would be at a height of

$$s\Big|_{t=1} = 100 - 16 = 84 \text{ feet}$$

and at time $t = 2$ it would be at a height of

$$s\Big|_{t=2} = 100 - 64 = 36 \text{ feet.}$$

Hence the average velocity of the object over the time interval would be

$$\frac{\Delta s}{\Delta t} = \frac{36 - 84}{2 - 1} = -48 \text{ feet/second.}$$

Note that the average velocity over this time interval is negative because we have taken the positive direction to be up. The average speed of the object, which is the absolute value of the velocity, would be 48 feet per second. To find the instantaneous velocity at time t, we compute

$$\begin{aligned}
\frac{ds}{dt} &= \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{(100 - 16(t + \Delta t)^2) - (100 - 16t^2)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{100 - 16(t^2 + 2t\Delta t + (\Delta t)^2) - 100 - 16t^2}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{-32t\Delta t - 16(\Delta t)^2}{\Delta t} \\
&= \lim_{\Delta t \to 0} (-32t - 16\Delta t) \\
&= -32t.
\end{aligned}$$

Hence the instantaneous velocity of the object at time $t = 1$ is

$$\left.\frac{ds}{dt}\right|_{t=1} = -32 \text{ feet/second}$$

and the instantaneous velocity at time $t = 2$ is

$$\left.\frac{ds}{dt}\right|_{t=2} = -64 \text{ feet/second.}$$

Although Leibniz seems to have thought of the expression $\frac{dy}{dx}$ as a ratio, we should think of $\frac{d}{dx}$ as the operation of differentiation, which, when applied to y, yields the derivative of

y with respect to x. In other words, we should not think of $\frac{dy}{dx}$ as a ratio, but as $\frac{d}{dx}(y)$. For example, if $y = x^3$, then, using an earlier result from this section, we might write

$$\frac{dy}{dx} = \frac{d}{dx}(x^3) = 3x^2.$$

The "prime" notation for a derivative is due, not to Newton, but to Joseph Louis Lagrange (1736-1813). Newton's notation, a dot above the dependent variable, represents a derivative with respect to time, denoted by t. For example, in the previous example we may write

$$\dot{s} = -32t$$

using Newton's notation. Because of its simplicity and the frequency with which derivatives with respect to time occur, this is often a useful notation and we will make extensive use of it when we study differential equations in Chapter 8.

Problems

1. Using (3.2.10), find the derivative of each of the following functions at the indicated point.

   (a) $f(x) = x^2 + 1$ at $x = 2$

   (b) $f(t) = \dfrac{1}{t}$ at $t = 1$

   (c) $g(x) = \dfrac{1}{x^2}$ at $x = 2$

   (d) $h(t) = \sqrt{t+1}$ at $t = 3$

   (e) $f(s) = \dfrac{1}{\sqrt{s}}$ at $s = 1$

   (f) $g(z) = (z+1)^2$ at $z = -1$

2. For each of the functions in Problem 1, find the best affine approximation for the function at the indicated point. Also, find the equation of the tangent line at that point and graph the function and its tangent line together.

3. Using (3.2.11), find the derivative of each of the following functions. Note any points where the given function is not differentiable.

   (a) $f(x) = 2x^2$

   (b) $g(x) = \dfrac{1}{x}$

   (c) $f(t) = \dfrac{1}{\sqrt{t}}$

   (d) $h(z) = \dfrac{1}{3z}$

   (e) $y(t) = t^2 + 4t$

   (f) $g(s) = 2s^3 - s^2$

4. Using your result from part (c) of Problem 3, find the best affine approximation to

$$f(t) = \frac{1}{\sqrt{t}}$$

   at $t = 4$. Use it to approximate $\dfrac{1}{\sqrt{3.98}}$.

5. Let $f(x) = ax^2 + bx + c$, where a, b, and c are constants. Show that $f'(x) = 2ax + b$. Does this result agree with your results in parts (a) and (e) of Problems 3?

6. Use your result from Problem 5 to find the best affine approximation to
$$f(x) = 3x^2 - 2x + 5$$
at $x = -2$.

7. Use your result from Problem 5 to find the best affine approximation to
$$g(t) = -2t^2 + 3t - 6$$
at $t = 3$.

8. Suppose f is a function with the properties that $f(0) = 0$ and
$$\lim_{t \to 0} \frac{f(t)}{t} = 1.$$
Show that $f'(0) = 1$.

9. Suppose g is a function with the properties that $g(0) = 0$ and g is $o(h)$. Show that $g'(0) = 0$.

10. Suppose f is a function with the properties that
$$f(s + t) = f(s)f(t)$$
for all numbers s and t and
$$\lim_{t \to 0} \frac{f(t) - 1}{t} = 1.$$
Show that $f'(t) = f(t)$.

11. Suppose $f(x) = \begin{cases} 3x^2, & \text{if } x < 0 \\ x^3, & \text{if } x \geq 0 \end{cases}$. Is f differentiable at $x = 0$? If it is, find $f'(0)$.

12. Suppose $g(t) = \begin{cases} 5t, & \text{if } t < 0 \\ 3t^2, & \text{if } t \geq 0 \end{cases}$. Is g differentiable at $t = 0$? If it is, find $g'(0)$.

13. Suppose $g(x) = \begin{cases} x^2 - 2x + 2, & \text{if } x \leq 1 \\ 4x - 3, & \text{if } x > 1 \end{cases}$. Is g differentiable at $x = 1$? If it is, find $g'(1)$.

14. For each of the following, find the derivative of the dependent variable with respect to the independent variable. Denote the derivative using Leibniz's notation.

    (a) $s = 2t^3$

    (b) $z = 2\sqrt{t}$

    (c) $q = s - \dfrac{2}{s}$

    (d) $t = x^4$

15. Find $\dfrac{d}{dx}(4x^2)$ and $\dfrac{d}{du}(3\sqrt{u-1})$.

16. An object is thrown vertically into the air from an initial height of 100 meters above the ground with an initial velocity of 10 meters per second. If s represents the height, in meters, of the object above the ground after t seconds and we ignore the effects of air resistance, then
$$s = 100 + 10t - 4.9t^2.$$
   (a) What is the average velocity over the time interval $[0, 2]$? Over $[0, 1]$? Over $[1, 2]$?
   (b) Find the velocity v of the object at time t. You may use Problem 5.
   (c) What is the velocity after 1 second? After 2 seconds?
   (d) When is $v = 0$? Is v positive or negative before this time? Is v positive or negative after this time?
   (e) What is the height of the object when $v = 0$? What is the significance of this height?
   (f) The rate of change of velocity is called *acceleration*. Find the acceleration of the object; that is, find $\frac{dv}{dt}$.
   (g) What is the significance of the fact that $\frac{dv}{dt}$ is a constant?

17. Find the rate of change of the area A of a circle with respect to its radius r.

18. Find the rate of change of the volume V of a sphere with respect to its radius r.

19. Find the rate of change of the area A of a square with respect to the length x of one of its sides.

20. Find the rate of change of the volume V of a cube with respect to the length x of one of its sides.