Method of Fundamental Solution (MFS)

Introduction

MFS is an efficient way to solve partial differential equations (PDEs). The basic idea of MFS is to approximate the solution \( u(x) \) by a linear combination of fundamental solutions of the problem:

\[
    u(x) \approx \sum_{j=1}^{m} c_j \phi_j(x - x_j)
\]

where \( c_j \) is the number of approximating functions each of which is centered at \( x_j \) and have coefficient \( c_j \).

Fundamental solution

The fundamental solution for Laplace and Helmholtz equation is the following:

\[
    \begin{align*}
        \phi(r) &= \frac{1}{r}, \quad \text{Laplace} \\
        \phi(r) &= \frac{1}{2\pi r}, \quad \text{Helmholtz}
    \end{align*}
\]

Numerical Results

**MFS combined with Fourier method**

MFS applied in 3D can be computationally costly. But for cylindrically symmetric objects, we can exploit the symmetry and further simplify the problem:

\[
    K_{0}(r, z', z') = \int_{0}^{2\pi} e^{i k r} \phi(r - r') dr' = \sum_{n=1}^{P} c_n e^{i k r} (r - r_n) \quad n = 1, 2, 3... P
\]

where \( (r, z) \) are the location of target point and MFS source ring in the \( rz \) plane, respectively. \( P \) is the number of Fourier modes. \( r = (r, z, 0), r_n = (r, z, 0) \) are the locations of the target point and \( n \)-th charge on the ring in a 3D cylindrical coordinate system, respectively. All the kernels can be evaluated once by the Fast Fourier Transform (FFT). The original 3D problem becomes \( P \) copies of independent 2D problems, one for each Fourier mode.

**Helmholtz equation for high-frequency acoustics**

We have implemented MFS for high-frequency Helmholtz problem. In figure 3(a), an acoustic wave is traveling in the \( +y \) direction and transmitted into the object and was measured on the plane \( y = 1.5 \) (see figure 3a). We can clearly see the shadow of the object. This method gives the error of order \( O(10^{-10}) \) even with high frequency \( k_x = 50, k_y = 75 \) (31 wavelengths in diameter) and it only takes 66 seconds to get the coefficients. Once the matrix is factorized, new incident waves can be solved in 2 seconds each. Figure 3b shows another wave traveling in the \( -z \) direction, met the objects, transmitted it and measured its intensity at plane \( z = -2 \) with frequency \( k_x = 10, k_y = 30 \).

**Full Maxwell equation**

The fundamental solution to Maxwell equation is just the electric and magnetic field generated by a single electric or magnetic dipole. Here we take the electric dipole as the MFS source point. The electric field \( E \) and magnetic field \( H \) at point \( r \) generated by a unit electric dipole located at \( r_0 \) and oriented along \( \tau \) is \([1]\)

\[
    \begin{align*}
        E &= \frac{3[R(R_\tau) - R_\tau]}{R} (1 - ikR) - \frac{k}{4\pi} \frac{1}{|R|} R \times (R \times \tau) e^{i k R/2} \\
        H &= \left( \frac{1}{|R|} - \frac{ik}{4\pi} \right) R \times (R \times \tau) e^{i k R/2}
    \end{align*}
\]

**Figure 4**

Figure 4 plots \( E_x \) in the plane \( x = 0 \), with dipole-source locating at \((10, 10, 0)\) and orienting at \((1, 1, 0)\) (a) \( k_x = 1 \) and the domain is perfect electric conductor, (b) \( k_x = 10 \) and the domain is dielectric. (c) \( k_x = 10 \) and the domain is perfect electric conductor, (d) \( k_x = 10 \) and the domain is dielectric.

**Future Work**

Periodization

MFS can also be used to deal with periodization. In the example shown in figure 5, there is a 2D square box with length 2 and centered at origin. We can add MFS source points outside of the box and impose the periodic conditions to get the unknown charge strengths of the MFS points. Figure 5 shows the plot of the solution inside of the box. We can clearly see that the solution satisfies the periodic condition. The error tested in this case is in the order \( O(10^{-10}) \) and it takes 0.06 seconds to get the coefficients. We will complete the periodization in 3D.

**Figure 6**

Figure 6: wave-scattering from periodic arrays of axisymmetric objects

References


Department of Physics & Astronomy, Department of Mathematics, Thayer School of Engineering, Dartmouth College