June 2018 Written Certification Exam

Algebra

1. Let L/K be a finite separable extension of fields, and let $\mathrm{N}_{L/K}$ denote the norm from L to K, and $\mathrm{Tr}_{L/K}$ the trace.

   (a) For a typical element $\alpha \in L$, describe how to compute $\mathrm{N}_{L/K}(\alpha)$ and $\mathrm{Tr}_{L/K}(\alpha)$.

   Next assuming in addition that L/K is Galois, give a reason based on Galois theory that the norm and trace take values in K.

   (b) With L/K an arbitrary finite separable extension, show that the trace is surjective. Hint: Since the trace is K-linear, it is enough to show the trace is not identically zero.

   (c) Now let $L = \mathbb{F}_{p^m}$, $K = \mathbb{F}_p$. Show that the norm $\mathrm{N}_{L/K}$ is surjective. Hint: It is enough to show that $\mathrm{N}_{L/K}(\mathbb{F}_{p^m}^\times) = \mathbb{F}_p^\times$.

2. Let ζ_n denote a primitive nth root of unity in $\mathbb{C}$. Let $E = \mathbb{Q}(\sqrt[3]{2}, \zeta_3)$ and $F = \mathbb{Q}(\zeta_7)$, and $L = EF$ their compositum.

   (a) Determine the degree $[L : \mathbb{Q}]$.

   (b) Determine the isomorphism class of $\mathrm{Gal}(L/\mathbb{Q})$ (in terms of standard groups).

   (c) Describe generators for $\mathrm{Gal}(L/\mathbb{Q})$.

3. Cyclic (Galois) extensions.

   (a) For your choice of base field K having characteristic 0, construct an extension L/K with $\mathrm{Gal}(L/K) \cong \mathbb{Z}/p\mathbb{Z}$ (p prime), and of course justify your assertions.

   (b) Let p be a prime, and $f(x) \in \mathbb{F}_p[x]$ irreducible of degree $n \geq 1$. Show that $\mathbb{F}_p(\xi)/\mathbb{F}_p$ is a Galois extension with Galois group cyclic of order n.

   (c) With $f(x) \in \mathbb{F}_p[x]$ irreducible of degree $n \geq 1$, show that f divides $x^{p^n} - x$ in $\mathbb{F}_p[x]$.

4. Compute the characteristic polynomial, minimal polynomial, and rational canonical form of the matrix

$$A = \begin{pmatrix} 1 & 2 & -4 & 4 \\ 2 & -1 & 4 & -8 \\ 1 & 0 & 1 & -2 \\ 0 & 1 & -2 & 3 \end{pmatrix}$$

5. Let V be an inner product space over $\mathbb{C}$ with inner product $\langle\, , \rangle$ that is $\mathbb{C}$-linear in the first variable (and conjugate linear in the second variable). Let $\phi \in \mathrm{End}_{\mathbb{C}}(V)$. Suppose that $\langle \phi(v), v \rangle \in \mathbb{R}$ for all $v \in V$. Show that ϕ is self-adjoint.

6. Let R be a ring (with 1), let M be a left R-module, and let $X, Y \subseteq M$ be R-submodules such that $M = X \oplus Y$.

   (a) Let $X' \subseteq M$ be an R-submodule. Show that $M = X' \oplus Y$ if and only if there exists $\phi \in \operatorname{Hom}_R(X, Y)$ such that $X' = (1 - \phi)(X)$. *[Hint: use a projection.]*

   (b) Let P, N, L be R-modules, suppose that P is a projective R-module, and let

   $$\psi \colon P \oplus N \to L$$

   be a surjective R-module homomorphism. Suppose that $\psi(N) = L$. Show that there exists a R-submodule $P' \subseteq \ker \psi$ such that $P' \oplus N = P \oplus N$ and $P' \simeq P$ as R-modules. *[Hint: Use part* (a)*. We have "realized ψ as a projection, after a change of coordinates".]*

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Analysis

1. Suppose that $(X, \mathcal{M}, \mu)$ is a measure space and that $f_n : X \to [0, \infty]$ is measurable for $n = 1, 2, 3, \ldots$. Suppose also that $f_{n+1}(x) \leq f_n(x)$ for all x and all n, while $f_n(x) \to f(x)$ for all x. If $f_1 \in \mathcal{L}^1(X)$, show that

$$\lim_{n\to\infty} \int_X f_n(x)\, d\mu(x) = \int_X f(x)\, d\mu(x). \tag{1}$$

   Also observe that the assertion is not true if we omit "$f_1 \in \mathcal{L}^1$".

2. (a) State the Maximum Modulus Principle.

   (b) Suppose that f is holomorphic on $D = B_1(0)$ and extends continuously to the boundary. Suppose that $|f(z)|$ is constant on the boundary $|z| = 1$. Show that either f is constant or f has a zero in D.

3. Suppose that f is an entire function such that

$$\lim_{z\to\infty} \left|\frac{f(z)}{z}\right| = 0. \tag{2}$$

   Show that f must be constant. (Hint: use the Cauchy Integral formula for the Derivatives.)

4. Let $(V, \|\cdot\|)$ be a normed vector space. Show that the formula

$$d(x,y) = \begin{cases} \|x-y\|, & \text{if } x = cy \text{ for some } c \in \mathbb{R} \\ \|x\| + \|y\|, & \text{otherwise} \end{cases}$$

   defines a metric on V. Is this metric induced by a norm on V? Why or why not?

5. Consider the space $C^1[-1,1]$ of continuous functions $f : [-1,1] \to \mathbb{R}$ with continuous derivative f'. Is the norm

$$\|f\| = \sqrt{\int_{-1}^{1} |x||f(x)|^2 + 3|f'(x)|^2\, dx}$$

   induced by an inner product? Justify your answer.

6. Let $(\mathcal{H}, (\cdot,\cdot))$ be a complex Hilbert space. Recall that an operator $S \in \text{Ł}(\mathcal{H})$ is called *positive* ($S \geq 0$) if $(S(x), x) \geq 0$ for all $x \in \mathcal{H}$. We say that S is a *contraction* if $\|S\| \leq 1$. Show that S is a contraction if and only if $I - S^*S \geq 0$.

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Topology

1. Compute the Lie bracket $[X, Y]$ of two vector fields $X = \frac{\partial}{\partial x} + z^{2018}\frac{\partial}{\partial z}$ and $Y = 3\frac{\partial}{\partial x} + y^{2019}\frac{\partial}{\partial y}$ on $\mathbb{R}^3$ with coordinates (x, y, z).

2. Let $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + 3x_3^2$ be a map $\mathbb{R}^3 \to \mathbb{R}$. Show that the solution of the equation $x_1^2 + x_2^2 + 3x_3^2 = 2018$ is a smooth orientable 2-dimensional manifold. You may want to use Euler characteristic to show that the surface is orientable.

3. **a** Let M^3 be a compact oriented smooth 3-dimensional manifold without boundary, and let N^4 be a compact smooth 4-dimensional oriented manifold without boundary. Let $\phi : M \to N$ and $\psi : N \to M$ be smooth maps and let ω be a smooth 4-form on N. Compute $\int_N (\phi \circ \psi)^* \omega$.

   b Let M^3 be a compact oriented smooth 3-dimensional manifold and let N^4 be a compact oriented smooth 4-dimensional manifold. Let ω be a smooth 3-form on M and $\phi : N \to M$ be a smooth map. Compute $\int_N d\big(\phi^* \omega\big)$.

4. Let $n > 1$. Show that any continuous map $f : \mathbb{R}\boldsymbol{P}^n \to \S^1$ from real projective n-space to the circle is homotopic to a constant map. [**Suggestion:** Consider the universal cover of $\S^1$.]

5. Let $f \in \mathbb{C}[X]$ be a polynomial with complex coefficients, regarded as a smooth function $f : \mathbb{C} \to \mathbb{C}$. Then f extends to a smooth map $f_+ : \S^2 \to \S^2$ on the one-point compactification $\S^2 = \mathbb{C}_+$ of $\mathbb{C}$ by defining $f_+(\infty) = \infty$. Show that the Brouwer degree of f_+ is just the degree of f as a polynomial.

6. Let P be the space obtained from the unit disk $\boldsymbol{D}^2$ by deleting the open balls B_1 of radius $\frac{1}{4}$ about $(\frac{1}{2}, 0)$ and B_2 of radius $\frac{1}{4}$ about $(-\frac{1}{2}, 0)$: that is, $P = \boldsymbol{D}^2 - (B_1 \cup B_2)$. Then P is a 2-manifold with boundary whose boundary consists of three connected components, the circles of radius $\frac{1}{4}$ about $(\frac{1}{2}, 0)$ and $(-\frac{1}{2}, 0)$ and the origin-centered unit circle. Let $X = P/\!\sim$ be the quotient space obtained from P by identifying these three circles together via the obvious homeomorphisms preserving the counterclockwise orientation: thus for each $\theta \in [0, 2\pi)$, we identify together the three points $(\cos\theta, \sin\theta)$, $(\frac{1}{2}, 0) + \frac{1}{4}(\cos\theta, \sin\theta)$, and $(-\frac{1}{2}, 0) + \frac{1}{4}(\cos\theta, \sin\theta)$. Determine the homology groups of the space X. [**Hint:** Give X a suitable CW structure.]

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Applied I

1. Analyze the well posedness of

$$u_t = u_{xxxx} - u_{xx} + u_x + u, \qquad u(x,0) = f(x). \tag{3}$$

While you may be able to ascertain the well-posedness of (3) immediately, you must verify your conclusion mathematically.

2. Consider the initial value problem for $x \in [0,1]$, $t \in [0,T]$:

$$u_t + au_x = 0, \tag{4}$$

$a \in \mathbb{R}$, with initial conditions $u(x,0) = \phi_k(x) = exp(2\pi ikx)$ and periodic boundaries (you do not have to consider boundary conditions in your analysis). We seek to approximate (x,t) on a set of grid points $u(x_j,t^n)$, for $j = 0,\cdots,J$, $n = 1,\cdots,$ with $x_j = j\Delta x$, $\Delta x = \frac{1}{J}$, $t^n = n\Delta t$, and Δt as the chosen (uniform) time step. Let the numerical solution of $u(x_j,t^n)$ be written as u_j^n.

Suppose we approximate the spatial derivative

$$-au_x(x_j,t^n) \approx \frac{a}{\Delta x}\left(-\frac{1}{12}u_{j-2}^n + \frac{2}{3}u_{j-1}^n - \frac{2}{3}u_{j+1}^n + \frac{1}{12}u_{j+2}^n\right). \tag{5}$$

(a) Without doing any work, can you predict the spatial order of accuracy for solving (4) independently of how $u_t(x_j,t^n)$ is approximated? Explain.

(b) Explicitly determine the spatial order of accuracy. Write your answer as $\mathcal{O}(\Delta x)^p$. Hint: To avoid cumbersome calculations, you will have to exploit the symmetries when determining p.

(c) What does the order of accuracy in the spatial derivative say about the long term behavior of the numerical solution? Specifically, what modified equation does (5) better approximate? You should talk about dispersion and diffusion in your answer.

(d) Suppose we approximate the temporal derivative as

$$u_t(x_j,t^n) \approx \frac{1}{2\Delta t}\left(u_j^{n+1} - u_j^{n-1}\right). \tag{6}$$

Combine (5) and (6) to write the numerical scheme for (4). What is the truncation error? Write your answer as $\mathcal{O}(\Delta x)^p, \mathcal{O}(\Delta t)^q$. (Assume we have an appropriate start up scheme to approximate $u(x_j,t^1)$.)

(e) What is the amplification matrix for this resulting scheme?

(f) Explain the Von Neumann condition for stability and describe what you would need to show in this case to satisfy the condition. Your answer should refer to the amplification matrix, but you should *not* try to find the explicit conditions on Δt for satisfying the Von Neumann condition – it will take too long! Does satisfying the Von Neumann condition guarantee stability? Why or why not?

3. Consider Burgers' equation for $x \in [-\pi, \pi]$:

$$u_t + (\frac{u^2}{2})_x = 0, \quad u(x, 0) = u_0(x) = \alpha + \beta \sin x,$$

with $\alpha, \beta > 0$. Show that the solution becomes discontinuous at $t = \frac{1}{\beta}$. Hint: First solve for u_x along the characteristics where $u(x, t) = u(\xi, 0) = u_0(\xi)$ is constant (Why?). The solution becomes discontinuous when u_x becomes infinite (Why?).

4. Consider a discrete time Markov chain with states $0, 1, \cdots, N$ whose matrix has elements

$$P_{ij} = \begin{cases} \mu_i, & j = i - 1 \\ \lambda_i, & j = i + 1, \ i, j = 0, 1, \cdots, N. \\ 1 - \lambda_i - \mu_i, & j = i, \\ 0, & |j - i| > 1. \end{cases}$$

Suppose that $\mu_0 = \lambda_0 = \mu_N = \lambda_N = 0$, and all other μ_i's and λ_i's are positive, and that the initial state of the process is k. Determine the absorption probabilities at 0 and N, respectively.

5. Assume there are N light bulbs in total. Each light bulb can be turned on or off inde- pende ntly. Periods of on and off for any one light bulb alternate, having exponential distributio ns with parameters λ and μ, respectively.

   (a) Write down the partial differential equation for the number of light bulbs that are of f at time t.

   (b) What is the steady-state solution as $t \to \infty$.

   (c) Calculate the mean number of light bulbs that are off at time t if initially there a re m light bulbs that are off at $t = 0$.

6. Consider a discrete time branching process with identical and independent distribution of offspring number, given by the probability generating function

$$\phi(s) = q + ps, \quad q, p > 0, \quad q + p = 1.$$

Assume there are initially N individuals.

   (a) Determine the ultimate extinction probability of the population.

   (b) Determine the probability distribution of the time T when the population first becomes extinct.

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Applied II

1. In 1940 the Russian applied mathematician A. Kolmogorov assumed there was a law for turbulent fluid flow relating the four quantities: l (length), E (energy, units of ML^2T^{-2}), ρ (density, mass per unit volume), and R (dissipation rate, energy per unit time per unit volume). Using this assumption and the Buckingham Pi Theorem, state the simple form the law must have. Show that there is a (famous!) scaling relation $E = \text{constant} \cdot l^\alpha$ when other parameters are held constant; give α.

2. Use the WKB method to find an approximation for the large eigenvalues and the corresponding eigenfunctions of the eigenvalue problem

$$y'' + \lambda e^{4x} y = 0, \text{ where } 0 < x < 1, \; y(0) = y(1) = 0 \text{ and } \lambda \gg 1.$$

3. (A model for fishery) A fish population $P(t)$ in a lake is harvested at a constant rate and it grows logistically. Conclude the dynamics of population growth is given by

$$\frac{dP}{dt} = rp(1 - \frac{P}{K}) - H,$$

where $H > 0$ is the per capita harvesting rate. Non-dimensionalize the system by introducing a dimensionless harvesting parameter h. Find all the equilibrium points and plot the phase line plots for different values of h. Show that a bifurcation occurs at a certain value h_c (the critical value of h), draw the bifurcation diagram and classify this bifurcation. Discuss the long term behavior of the fish population for $h < h_c$ and $h > h_c$.

4. Consider the problem:

$$\min_{\boldsymbol{x} \in \mathbb{R}^d} ||\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}||_\infty,$$

where $\boldsymbol{A} := [\boldsymbol{a}_1, \cdots, \boldsymbol{a}_m]^\top \in \mathbb{R}^{m \times d}$ and $\boldsymbol{b} := [b_1, \cdots, b_m]^\top \in \mathbb{R}^m$.

   (a) Is the problem convex? Explain.

   (b) Write an equivalent linear programming problem.

   (c) Derive a dual problem.

   (d) Explain a weak duality and a strong duality. State a strong duality condition for the problem.

5. (a) Write down the definition of the proximal operator of a function $\phi : \mathbb{R}^d \to (-\infty, \infty]$.

(b) Derive the proximal operator of $\phi(\boldsymbol{x}) = ||\boldsymbol{W}\boldsymbol{x}||_1$ where $\boldsymbol{W} := \mathrm{diag}(\boldsymbol{w}) \in \mathbb{R}^{d\times d}$ is a diagonal matrix with positive diagonal elements $\boldsymbol{w} := (w_1, \cdots, w_d)^\top \in \mathbb{R}^d_{++}$.

(c) Describe a proximal gradient method for solving the following problem:

$$\min_{\boldsymbol{x}\in\mathbb{R}^d} \{f(\boldsymbol{x}) + ||\boldsymbol{W}\boldsymbol{x}||_1\}$$

where $f(\boldsymbol{x}) = \frac{1}{2}||\boldsymbol{y} - \boldsymbol{A}\boldsymbol{x}||_2^2$.

(d) Write down the optimality condition of the problem.

6. Consider the problem:

$$\begin{aligned}\min_{x_1,x_2\in\mathbb{R}} \ & x_1^2 + \frac{1}{2}x_2^2 + 2x_1x_2 - 4x_1 - 5x_2 \\ \text{subject to } & 2x_1 + 2x_2 \le 1.\end{aligned}$$

(a) Derive a dual problem of the given (primal) problem.

(b) Find the optimal solutions of both the dual and primal problems.