September 2017 Written Certification Exam

Algebra

Your Name:

1. Let R be a (commutative) domain with finitely many elements. Show that R is a field.

2. Let R be a commutative ring (with 1).
   (a) Let $S = R \times R$, and let $M = \{0\} \times R \subseteq S$. Is M a projective S-module?
   (b) Let M be a projective R-module. Is $\mathrm{Hom}_R(M, R)$ a projective R-module?
   (c) Let k be a field and let $R = k[t]$ be the polynomial ring over k in the variable t. Let $M = R^3$ with standard basis e_1, e_2, e_3 and let N be the quotient of R^3 by the relations

$$\begin{aligned}(1+t)e_1 + te_2 + e_3 &= 0\\ te_1 + e_3 &= 0\\ t^2e_1 - te_2 &= 0\end{aligned}$$

   Show that N is a torsion R-module and compute the invariant factors of N as an R-module. Is N projective as an R-module?

3. Let $n \geq 1$ and let $A \in \mathrm{M}_n(\mathbb{R})$ be a symmetric matrix. Consider the $\mathbb{R}$-linear map

$$\begin{aligned}T : \mathrm{M}_n(\mathbb{R}) &\to \mathrm{M}_n(\mathbb{R})\\ X &\mapsto AX - XA.\end{aligned}$$

   (a) For $n = 2$ and $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ compute the characteristic polynomial of T.
   (b) Suppose A is diagonal (and n is arbitrary). Show that T is diagonalizable and compute the eigenvalues of T.
   (c) Show that T is diagonalizable without the assumption that A is diagonal.

4. Polynomials and factoring.
   (a) Consider the polynomial $f(x) = x^{255} - 1 \in \mathbb{Q}[x]$. Describe the factorization of f as a product of irreducibles in $\mathbb{Q}[x]$, and indicate the degrees of the irreducible factors. It may be useful to note that $255 = 3 \cdot 5 \cdot 17 = 2^8 - 1$.
   (b) Let $F = \mathbb{F}_{16}$, the finite field with 16 elements. For each of the polynomials below, determine how many roots it has in F.
      i. $x^5 - 1$.

ii. $x^8 - 1$.

iii. $x^{15} - 1$.

iv. $x^{17} - 1$.

5. Let L/K be Galois with $[L : K] = p^2q$ with primes $q < p$ satisfying $q \nmid (p^2 - 1)$.

   (a) Show there exist fields $K \subset E, F \subset L$ with $[E : K] = p^2$ and $[F : K] = q$.

   (b) Show that E/K and F/K are Galois extensions.

   (c) Prove that $\mathrm{Gal}(L/K)$ is abelian.

6. Let $\varphi : R \to S$ be a surjective homomorphism between commutative rings with identity.

   (a) For an ideal $J \subset S$, characterize $\varphi(\varphi^{-1}(J))$.

   (b) For an ideal $I \subset R$, characterize $\varphi^{-1}(\varphi(I))$.

   (c) State the correspondence theorem for ideals as it applies to the rings R and S.

   (d) Proof or counterexample: If R is a Noetherian commutative ring with identity, and $\varphi : R \to S$ is a surjective ring homomorphism, then S is Noetherian.

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Analysis

Your Name:

1. (a) State the Maximum Modulus Principle.
   (b) Let Ω be a region in the complex plane, $a \in \Omega$ and $r > 0$ such that $\overline{D_r(a)} \subset \Omega$. Suppose that $f \in H(\Omega)$ has no zero in $D_r(a)$. Prove that
   $$|f(a)| \geq \min_{\theta} |f(a + re^{i\theta})|.$$
   Hint: Consider $1/f$.

2. Let $(X, \mathfrak{M}, \mu)$ be a measured space and $f \in \mathcal{L}^1(X, \mathfrak{M}, \mu)$.
   (a) Prove that the sequence of functions
   $$f_n = \inf(|f|, n)$$
   converges to $|f|$ in $L^1(X, \mathfrak{M}, \mu)$.
   (b) Deduce, for every $\varepsilon > 0$, the existence of $\delta > 0$ such that
   $$\forall A \in \mathfrak{M}\,,\ \mu(A) \leq \delta \Rightarrow \int_A |f|\, d\mu \leq \varepsilon.$$

3. (a) State Morera's Theorem.
   (b) Let f be a continuous function on $D = D_1(0)$. Assume that f is holomorphic in $D^+ = \{z \in D \,:\, \operatorname{Im}(z) > 0\}$ and $D^- = \{z \in D \,:\, \operatorname{Im}(z) < 0\}$. Prove that f is holomorphic in $D_1(0)$.

4. Suppose that H and K are Hilbert spaces and $T : H \to K$ is a bounded linear operator. Show that there is a bounded linear operator $T^* : K \to H$ such that
   $$\big(Th \mid k\big)_K = \big(h \mid T^*k\big)_H \quad \text{for all } h \in H \text{ and } k \in K, \tag{1}$$
   and that $\|T^*\| = \|T\|$.

5. Let E and Y be closed subspaces of a normed vector space X. Suppose that E is finite dimensional. Show that $E + Y = \{\, x + y : x \in E \text{ and } y \in Y \,\}$ is closed in X. (Hint: consider the quotient map $q : X \to X/Y$.)

6. Let X be a normed vector space. Let $B_\delta(x) = \{\, y \in X : \|x - y\| < \delta \,\}$.
   (a) Show that $B_\delta(x) - B_\delta(x) = B_{2\delta}(0)$.
   (b) Suppose that $T : X \to X$ is linear and that $T(B_1(0))$ has interior. Show that T is open.
   (c) In part (b), suppose that T is bijective. Show that T^{-1} is bounded.

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Topology

Your Name:

1. Let M be a smooth manifold that is topologically $\mathbb{R}^4$ (with coordinates (x, y, z, t)) equipped with the smooth atlas consisting of all the charts $\tilde{\phi}$ that are smoothly compatible to $\phi(x, y, z, t) = (x^3, y^{\frac{1}{5}}, z, t)$.

   Let N be a smooth manifold that is topologically $\mathbb{R}^4$ (with coordinates (x, y, z, t)) equipped with the smooth atlas consisting of all the charts $\tilde{\psi}$ that are smoothly compatible to $\psi(x, y, z, t) = (x, y, z + t, z - t)$.

   Are these two manifolds diffeomorphic? Prove your answer.

2. (a) What is a critical value of a smooth map?
   (b) What does it mean that a subset of a manifold has measure zero?
   (c) State Sard's Theorem.
   (d) Do there exist **smooth surjective** maps $\gamma : [0, 1] \to [0, 1] \times [0, 1]$?
   Explain your answer.

3. Let M^m be a smooth m-dimensional manifold with boundary, and let $S \subset M$ be an embedded compact oriented k-dimensional submanifold without boundary. Let ω be a closed k-form on M. Show that if $\int_S \omega \neq 0$, then both of the following statements are true:

   (a) ω is not exact;
   (b) S is not the boundary of a compact oriented smooth submanifold in M with boundary.

4. Use Van Kampen's Theorem to prove that $\pi_1(S^n) = 0$ for $n \geq 2$.

5. Let $q : S^3 \to \mathbb{R}P^3$ be the quotient map that identifies antipodal points $x \sim -x$ of S^3. $\mathbb{R}P^3$ has a standard CW structure with a single 0-cell, 1-cell, 2-cell and 3-cell. If X^2 is the 2-skeleton of this CW complex, prove that the inclusion $i : X^2 \to \mathbb{R}P^3$ does not lift to a map $j : X^2 \to S^3$ with $i = q \circ j$.

6. Let M_g be a genus g surface embedded in $\mathbb{R}^3$, and A the bounded connected open subset of $\mathbb{R}^3$ with boundary M_g. Compute the homology groups $H_n(X; \mathbb{Z})$ of the complement $X = \mathbb{R}^3 \setminus A$.