June 2017 Written Certification Exam

Algebra

Your Name:

1. Let p be an odd prime. Let

$$\mathrm{O}_2(\mathbb{F}_p) = \left\{ A \in \mathrm{GL}_2(\mathbb{F}_p) : AA^t = A^tA = I \right\}.$$

Then $\mathrm{O}_2(\mathbb{F}_p) \leq \mathrm{GL}_2(\mathbb{F}_p)$ is a subgroup, called the *(standard) orthogonal group* in $\mathrm{GL}_2(\mathbb{F}_p)$.

   (a) Let $p = 5$. Show that $\# \mathrm{O}_2(\mathbb{F}_p) = 8$ and classify $\mathrm{O}_2(\mathbb{F}_p)$ up to isomorphism (i.e., give it a more familiar name).

   (b) For p an arbitrary odd prime, show that $\mathrm{O}_2(\mathbb{F}_p)$ always has a nontrivial normal subgroup of index 2 and write down the corresponding exact sequence of groups. Does this sequence split?

2. Let R be a commutative ring, let M be an R-module, and let $\phi : M \to M$ be an R-module homomorphism.

   (a) Show that $\phi^2 = 0$ if and only if $\phi(M) \subseteq \ker \phi$.

   (b) Suppose $R = F$ is a field and $M = V$ is finite-dimensional as an F-vector space. Show that there is an ordered basis β for V such that $[\phi]_\beta$ has the block form

$$\begin{pmatrix} O & A \\ O & O \end{pmatrix}$$

   i.e., has zeros in all blocks except possibly the upper right-hand corner.

3. Let V, W be finite-dimensional vector spaces over a field F, and let $\phi : V \to W$ be an F-linear map. Let $\phi^* : W^* \to V^*$ be the dual map. Show that ϕ is surjective if and only if ϕ^* is injective.

4. Polynomials

   (a) Characterize (and determine the number of) all proper, non-trivial ideals of the quotient rings $\mathbb{Q}[x]/(x^4 - 1)$ and $\mathbb{Q}[x]/(x^4 + 1)$, where (f) denotes the ideal of $\mathbb{Q}[x]$ generated by f.

   (b) Characterize the structure of the quotient ring $\mathbb{Q}[x, y]/(x^2 + 1, x^2 + y^4)$ by showing it is isomorphic to something involving simple rings (in both senses of the word simple).

5. Let E be the splitting field of $x^5 - 3$ over $\mathbb{Q}$, F the splitting field of $x^5 - 7$ over $\mathbb{Q}$, and put $L = EF$, their compositum. **Note:** You may assume that $x^5 - 7$ has no roots in E and $x^5 - 3$ has no roots in F.

   (a) Determine the degree $[L : \mathbb{Q}]$.

   (b) Determine the isomorphism class of $\mathrm{Gal}(L/F)$.

   (c) Determine the number of Sylow p-subgroups for each prime p dividing the order of $\mathrm{Gal}(L/\mathbb{Q})$.

6. Let $i = \sqrt{-1} \in \mathbb{C}$, $\zeta_5 \in \mathbb{C}$ a primitive 5th root of unity, and put $E = \mathbb{Q}(\zeta_5)$.

   (a) Show that $i \notin E$.

   (b) Let $L = E(i)$. Consider the norm, $N_{L/E}$, from L to E. Show that the image of the norm consists of those elements in E which can be written as the sum of two squares in E.

   (c) Determine the isomorphism class of $\mathrm{Gal}(L/\mathbb{Q})$.

   (d) Determine whether a regular 20-gon is constructible by straightedge and compass.

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Analysis

Your Name:

1. Let $p \geq 1$ be an integer and f a holomorphic function on $D_1(0)$ such that
   (i) $|f(z)| \leq |z|^p$ for all z with $|z| < 1$;
   (ii) f has a zero of order $\geq p$ at 0.

   Assume further the existence of $a \neq 0$ such that $f(a) = a^p$. What can be said of f?

2. Let $(X, \mathfrak{M}, \mu)$ be a measured space, $(Y, \mathfrak{N})$ a measurable space and $f : X \to Y$ a measurable function. Define for $S \in \mathfrak{N}$:

$$\nu(S) := \mu(f^{-1}(S)).$$

   (a) Verify that ν is a well-defined measure on $(Y, \mathfrak{N})$.
   (b) Let $u : Y \longrightarrow \mathbb{R}$ be a measurable function. Prove that

$$\int_Y u\, d\nu = \int_X u \circ f\, d\mu.$$

3. (a) Give an example of a sequence $\{f_n\}_{n\geq 1}$ of measurable functions such that $f_n \xrightarrow{L^1} f$ but f_n does not converge to f almost everywhere.

   (b,c) For parts (b) and (c), let $\{f_n\}_{n\geq 1}$ and f be functions on a measured space X such that for all n,

$$\|f_n - f\|_1 \leq 3^{-n}.$$

   (b) Let $E_n = \{x \in X \ : \ |f_n(x) - f(x)| \geq \frac{1}{2^n}\}$ and $G_k = \bigcup_{n\geq k} E_n$. Prove that

$$\mu(G_k) \leq \frac{2^k}{3^{k-1}}.$$

   (c) Deduce that $f_n \longrightarrow f$ almost everywhere.

4. Let H be a Hilbert space and T a normal operator on H.
   (a) Show that for all $h \in H$ we have $\|Th\| = \|T^*h\|$.
   (b) Show that if v is an eigenvector for T, then v is an eigenvector for T^*.

(c) Show that if v and w are eigenvectors for T with eigenvalues λ and μ, respectively, then $v \perp w$ if $\lambda \neq \mu$.

5. Let X be a normed vector space such that X^* is separable. Show that X is separable. (I suggest letting $\{\phi_n\}$ be a countable dense subset of X^* and choose $x_n \in X$ such that $\|x_n\| = 1$ and $\phi_n(x_n) \geq \frac{1}{2}\|\phi_n\|$. Consider $\operatorname{span}(\{x_n\})$.)

6. Let $\mathcal{F} \subset C([0,1])$ be the collection of continuous functions such that $f'(x)$ exists and satisfies $|f'(x)| \leq 2$ for all $x \in (0,1)$. Let $\mathcal{F}_0 = \{\, f \in \mathcal{F} : |f(0)| \leq 3 \,\}$.
   (a) Show that $\mathcal{F}$ is equicontinuous on $[0,1]$.
   (b) Explain why the closure of $\mathcal{F}$ is not compact, while the closure of $\mathcal{F}_0$ is.

June 2017 Written Certification Exam

Topology

Your Name:

1. Prove that a nonempty smooth manifold M of dimension m cannot be diffeomorphic to an n-dimensional manifold, unless $m = n$.

2. Which of the following manifolds are parallelizable? (Recall that an m-dimensional manifold M is parallelizable if it admits m vector fields whose values at every point of M are linearly independent.) Explain your answers.

   a. S^2;

   b. S^2 minus a point;

   c. S^2 minus two points;

   d. $SO(n, \mathbb{R})$;

   e. the Klein Bottle;

   f. An oriented compact surface of genus 4 with no boundary.

3. Let M_1, M_2 be oriented, compact k-dimensional manifolds with no boundary, and let $\phi_i : M_i \to N, i = 1, 2$, be smooth maps of manifolds. Assume moreover, that there exists a $(k+1)$-dimensional oriented compact manifold Σ whose oriented boundary is $\partial\Sigma = M_1 \sqcup -M_2$ and a smooth map $F : \Sigma \to N$ such that $F|_{\partial\Sigma} = \phi_1 \sqcup \phi_2$. Let ω be a closed k-form on N. Prove that $\displaystyle\int_{M_1} \phi_1^*\omega = \int_{M_2} \phi_2^*\omega$.

4. $\mathbb{R}P^n$ has a standard CW structure with a single k-cell for $k = 0, 1, \ldots, n$. Prove that there is no retract from this CW complex onto its 1-skeleton.

5. Use a Mayer-Vietoris sequence to prove the isomorphism of reduced homology groups $\widetilde{H}_k(S^n) \cong \widetilde{H}_{k-1}(S^{n-1})$ for $n \geq 1$, $k \geq 1$.

6. Identify S^1 with the complex unit circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. Let $X = S^1 \times [0, 1]/\sim$ be the quotient space obtained from the cylinder $S^1 \times [0, 1]$ by identifying points $(z, 0) \sim (iz, 0) \sim (-z, 0) \sim (-iz, 0)$ for any $z \in S^1$, and likewise $(z, 1) \sim (iz, 1) \sim (-z, 1) \sim (-iz, 1)$ for any $z \in S^1$ at the other boundary component. Here $i = \sqrt{-1}$.

   Compute the homology groups $H_n(X; \mathbb{Z})$.

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Applied

Your Name:

1. Let $g : \mathbb{R} \to \mathbb{R}$ be a smooth function having a fixed point x^* and the fixed point iteration be defined by $x_{k+1} = g(x_k)$, $k = 1, \ldots$.

   (a) Sketch a proof of the result:
   If $|g'(x^*)| < 1$, then the iteration is locally convergent and if $|g'(x^*)| > 1$ the fixed point iteration diverges for any starting point other than x^*.

   (b) Use your results from part (a) to determine the rate of convergence.

   (c) Use Taylor's theorem to deduce the condition under which the iteration converges quadratically.

   (d) Is Newton's method for finding a zero of a smooth function $f : \mathbb{R} \to \mathbb{R}$ an example of such a fixed point iteration scheme? If so, what is the function g in this case? If not, then explain why not.

   (e) For the following two functions determine whether the fixed point iteration converges locally to either of the real roots of $x^4 = 1/16$:

      i. $g_1(x) = x + x^4 - 1/16$.

      ii. $g_2(x) = 1 + x - 16x^4$

2. Let A_n be the 2×2 matrix given by

$$A_n = \begin{pmatrix} 1 & 2 \\ 2 & 4 + 1/n^2 \end{pmatrix}$$

   (a) Find A_n^{-1} and the condition number of A_n. (Use the one norm to calculate the condition number).

   (b) Let $n = 100$. Use Gaussian elimination without pivoting to solve $A_{100} \begin{pmatrix} x \\ y \end{pmatrix} = b$ using 5 significant figures at all stages of the calculation when

$$b = (1, 2 - 1/n^2)^T.$$

   (c) Repeat part (b) using 2 significant figures in the calculation.

   (d) Explain the answers in parts (b) and (c).

3. Consider the well-posed initial value problem $u_t = f(t,u)$, $t > 0$, with $u(0) = u_0$. Suppose we use the following scheme to solve this IVP:

$$y_{n+1} = \frac{1}{2}(y_n + y_{n-1}) + \frac{h}{4}(4f_{n+1} - f_n + 3f_{n-1}),$$

where h is the (fixed) time step.

   (a) Find the order of accuracy of this scheme and the leading term of the truncation error.

   (b) Does this scheme satisfy the *root condition*? Explain. Is this scheme *zero stable*? Explain.

   (c) Define absolute stability. What is the difference between zero and absolute stability? Derive the equation for the absolute stability region for this scheme. Your solution should be an explicit expression for time step h. You do not have to solve it.

4. Consider a random walk on the integers such that the transition probabilities $p_{i,i+1} = p$, $p_{i,i-1} = q$ for all integers i ($0 < p < 1$, $p + q = 1$).

   (a) Determine the n-step transition probability $p_{00}^{(n)}$.

   (b) Find the generating function of $u_n = p_{00}^{(n)}$, i.e. $P(x) = \sum_{n=0}^{\infty} u_n x^n$.

   (c) Determine the generating function of the recurrence time from state 0 to state 0.

   (d) What is the probability of eventual return to the origin?

5. Consider a *critical* homogeneous birth-and-death process (birth rate λ = death rate μ), starting with one single individual.

   (a) Write down the Kolmogorov backward equation for the probability $p_{1j}(t)$ that the population size transitions from 1 to j at time t.

   (b) Using (a), derive the backward recursive equation for the probability generating function $P(x,t)$ for the population size $X(t)$ distribution at t. And solve $P(x,t)$.

   (c) Find the probability $p_0(t)$ that the population becomes extinct by time t.

   (d) What is the mean extinction time?

   (e) What is the probability that the population size *ever* reaches n?

6. Let X be a nonnegative integer-valued random variable with probability generating function $f(x) = \sum_{n=0}^{\infty} a_n x^n$. After observing X, then conduct X binomial trials with probability p of success. Let Y denote the resulting number of successes.

   (a) Determine the generating function of Y.

(b) Determine the generating function of X given that $Y = X$.

(c) Suppose that for every $p (0 < p < 1)$ the probability generating functions in (a) and (b) coincide. Prove that the distribution of X is Poisson, $f(x) = e^{\lambda(x-1)}$ for some $\lambda > 0$.