June 2016 Written Certification Exam

Algebra

Your Name:

1. Let V be a 3-dimensional $\mathbb{Q}$-vector space, and let $T : V \to V$ be a linear operator that has eigenvalues 1 and 2 but is not diagonalizable.
   (a) What are the possible rational canonical forms of T?
   (b) What are the possible Jordan canonical forms of the operator $\mathrm{Id} \otimes T : \mathbb{C} \otimes_{\mathbb{Q}} V \to \mathbb{C} \otimes_{\mathbb{Q}} V$ on the complexification?

2. Let G be a finite p-group, and let $H \triangleleft G$ be a nontrivial normal subgroup.
   (a) Show that $H \cap Z(G) \neq \{1\}$, where $Z(G)$ is the center of G.
   (b) Show that the hypothesis that H is normal in G cannot be omitted above.

3. Let A and B be finitely generated abelian groups. Let $x \in A$ and $y \in B$ be elements of infinite order. Prove that $x \otimes y$ is a nonzero element of $A \otimes_{\mathbb{Z}} B$.

4. Let k be a field, and $k[x, y]$ a polynomial ring in two variables with coefficients in k.
   (a) The polynomials $y^2 - (x^3 - x)$ and $y^2 - x^3$ are both irreducible in $k[x, y]$. Pick one of these polynomials and prove that it is irreducible. In parts (b) and (c), you may assume both polynomials are irreducible in $k[x, y]$.
   (b) Show that the quotient ring $k[x, y]/(y^2 - x^3)$ is a Noetherian, integral domain.
   (c) Show that the principal ideal $(y^2 - x^3)$ is not maximal in $k[x, y]$, but $(y^2 - x^3)$ is a maximal ideal in $k(x)[y]$.

5. Consider a tower of fields $K \subset F \subset E$.
   (a) Show that the extension E/K is algebraic if and only if E/F and F/K are algebraic.
   (b) Suppose that E/K is an algebraic extension of fields, and F/K is any extension of K (and E, F lie in some common field). Show that the extension EF/F is algebraic.

6. Let L be the splitting field of $x^4 - 2$ over $\mathbb{Q}$.

   (a) Determine the degree $[L : \mathbb{Q}]$, and generators for $\mathrm{Gal}(L/\mathbb{Q})$. Also determine the isomorphism class of this Galois group (e.g., a standard group like $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$).

   (b) Write down the complete lattice of subgroups of the Galois group, and the corresponding lattice of intermediate fields between L and $\mathbb{Q}$. Identify a majority of the intermediate fields, both as an appropriate fixed field, and with generator(s) over $\mathbb{Q}$.

June 2016 Written Certification Exam

Analysis

Your Name:

1. Let $(X, \mathcal{M}, \mu)$ be a measure space, and suppose that $f_n : X \to \mathbb{R}$ is a measurable function for each $n \geq 1$. Further suppose that

$$\sup_{n\geq 1} \{f_n\} \in L^1(X, \mathcal{M}, \mu).$$

Show that

$$\int_X \limsup f_n \, d\mu \geq \limsup \int_X f_n \, d\mu.$$

2. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. Let $\mathbb{N}$ be the set of natural numbers, let $\mathcal{P}(\mathbb{N})$ be the power set of $\mathbb{N}$, and let ν be the counting measure. Consider the product measure space $(\mathbb{N} \times X, \mathcal{P}(\mathbb{N}) \otimes \mathcal{M}, \nu \times \mu)$. (Here $\mathcal{P}(\mathbb{N}) \otimes \mathcal{M}$ is the product σ-algebra.)

   (a) Let $E \subset (\mathbb{N} \times X)$ and for $n \in \mathbb{N}$, let

   $$E_n = \{x \in X : (n, x) \in E\}.$$

   Show that $E \in \mathcal{P}(\mathbb{N}) \otimes \mathcal{M}$ if and only if $E_n \in \mathcal{M}$ for every $n \in \mathbb{N}$.

   (b) Given a function $F : \mathbb{N} \times X \to \mathbb{R}$ and $n \in \mathbb{N}$, define $F_n : X \to \mathbb{R}$ by $F_n(x) := F(n, x)$. Show that F is $(\mathcal{P}(\mathbb{N}) \otimes \mathcal{M})$-measurable if and only if each function F_n, $n \in \mathbb{N}$, is $\mathcal{M}$-measurable.

   (c) Interpret Tonelli's Theorem in this setting. (Recall that Tonelli's Theorem is the part of the Fubini-Tonelli Theorem concerning nonnegative functions.) More precisely, indicate what familiar result is equivalent to the equality of the iterated integrals.

3. For each of the following, explain why there *cannot* be a function $f : \mathbb{C} \to \mathbb{C}$ with the stated properties.

   (a) f is an entire function and $\int_C f(z) \, dz = 5$, where C is the positively oriented circle $|z| = 1$.

(b) f is entire, $f(yi) = yi$ for $0 \le y \le 1$ and $f(7+2i) = 2i$.

(c) f is entire and $|f(x+yi)| = e^{-(x^4+y^4)}$ for all $x + yi \in \mathbb{C}$.

(d) f is entire, f has a zero of order 5 at the origin and $\int_C f(\frac{1}{z})\,dz = 2\pi i$, where C is the positively-oriented circle $|z| = 1$.

4. Let X and Y be Banach spaces. Suppose that $T : X \to Y$ is a linear map such that if (x_n) is a sequence converging weakly to zero in X, then $\big(T(x_n)\big)$ converges weakly to zero in Y. Show that T is bounded.

5. Let $A = C([0,1])$ with the uniform norm. Let $K \in C([0,1] \times [0,1])$. For each $f \in A$ and $x \in [0,1]$, define

$$T(f)(x) = \int_0^1 K(x,y) f(y)\,dy.$$

(a) Show that $T(f) \in A$.

(b) Show that $T \in \mathcal{L}(A)$.

(c) Let $B = \{\, f \in A : \|f\|_\infty \le 1 \,\}$. Show that $\overline{T(B)}$ is compact in A.

6. Let X be a compact metric space. Suppose that $f_n : X \to \mathbb{R}$ is continuous for each $n \ge 1$, and that for each $x \in X$, $f_{n+1}(x) \le f_n(x)$. Show that if $f_n \to 0$ pointwise on X, then $f_n \to 0$ uniformly on X.

June 2016 Written Certification Exam

Topology

Your Name:

1. Let M^m, N^m be smooth manifolds of the same dimension, and suppose that M^m is compact. Show that if $f : M^m \to N^m$ is a submersion, it is a covering.

2. Let $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$, and consider the map $r : S^n \to S^n$ defined by $r(x) = -x$. Show that r is orientation-reversing if and only if n is even.

3. Let $(x_1, \ldots, x_n)$ be the standard coordinates on $\mathbb{R}^n$ and $(y_1, \ldots, y_{n+1})$ the standard coordinates on $\mathbb{R}^{n+1}$. Let $f : \mathbb{R}^n \to \mathbb{R}^{n+1}$ be the map given by

$$f(x_1, \ldots, x_n) = \left(x_1, \ldots, x_n, \sum_{i=1}^{n} x_i^2 \right).$$

Compute the induced metric f^*g on $\mathbb{R}^n$, where $g = \sum_{j=1}^{n+1} dy_j^2$ is the standard metric on $\mathbb{R}^{n+1}$. Your answer should be expressed in the form $\sum_{i,j} g_{ij}\, dx_i\, dx_j$ where the g_{ij} are smooth real-valued functions on $\mathbb{R}^n$.

4. Show that if A is a deformation retract of X and $x_0 \in A$, then the induced map $i_* : \pi_1(A, x_0) \to \pi_1(X, x_0)$ is an isomorphism, where $i : A \to X$ is the inclusion map.

5. Let X be the topological space obtained from two copies of the 2-sphere S^2 and one copy of the circle $S^1 = \{z \in \mathbb{C} : \|z\| = 1\}$ by identifying the point $1 \in S^1$ with the north pole of the first 2-sphere and the point $-1 \in S^1$ with the north pole of the second 2-sphere. Draw the universal cover of X.

6. Use the Mayer-Vietoris sequence to compute all the homology groups of the space X obtained from the torus $S^1 \times S^1$ by attaching a Moebius band M via the homeomorphism from the boundary circle of M to the circle $S^1 \times \{x_0\} \subset S^1 \times S^1$.