September 2015 Written Certification Exam
Algebra

1. Let $M = \mathbb{Z}^n$ and denote by $pM = (p\mathbb{Z})^n$. Suppose that L is a submodule of M with $pM \subset L \subseteq M$.

   (a) Show that L is a free $\mathbb{Z}$-module of rank n.

   (b) Show that index $[M : L]$ is finite, and in terms of the index describe the invariant factors (or elementary divisors) of L in M as a $\mathbb{Z}$-module.

   (c) Now we fix $n = 2$: Let $pM = (p\mathbb{Z})^2 \subsetneq L \subsetneq \mathbb{Z}^2$. Count the number of possible submodules L.

2. Let F be a field and $f \in F[x]$ a polynomial with $\deg(f) = n \geq 1$, and suppose that $f = f_1^{e_1} \cdots f_t^{e_t}$ is the (unique) factorization of f into a product of irreducibles (i.e., f_i irreducible, and $\gcd(f_i, f_j) = 1$ for $i \neq j$). Let

$$S = \{A \in M_n(F) \mid \chi_A = f\},$$

   where χ_A denotes the characteristic polynomial of A.

   (a) Show that S is the union of a finite number of similarity (conjugacy) classes.

   (b) Show that the number of similarity classes in S is one if and only if $e_1 = e_2 = \cdots = e_t = 1$.

3. (a) Let G be a finite group, and let H be a subgroup with index $[G : H] = n \geq 2$. Show that if G is a simple group, then there exists an injective group homomorphism $\varphi : G \to S_n$ where S_n is the symmetric group on n letters.

   (b) Show that no group of order $96 = 3 \cdot 2^5$ is simple.

4. Let G be a finite group. Let $G' \leq G$ be the subgroup generated by all elements of the form $\sigma\tau\sigma^{-1}\tau^{-1}$ for $\sigma, \tau \in G$. (We call G' the *commutator* subgroup of G.)

   (a) Show that $G' \trianglelefteq G$ is normal and that G' is the smallest normal subgroup of G such that G/G' is abelian.

   (b) Now suppose $K \supseteq F$ is a Galois extension with Galois group G. Show that the fixed field of G' corresponds to the maximal subextension $K \supseteq M \supseteq F$ such that M/F is Galois with abelian Galois group.

5. Let $f(X) = (X^4 - 2X^2 - 1)(X^2 - 2)(X^2 + 1)$.

   (a) Show that $g(X) = X^4 - 2X^2 - 1$ is irreducible over $\mathbb{Q}$.

   (b) Exhibit a splitting field $K = K_f$ for f and show that it is equal to the splitting field for g.

   (c) Show that $\mathrm{Gal}(f)$ is nonabelian.

6. Let F be a field with $\operatorname{char} F \neq 2$. Let $a \in F^\times \setminus F^{\times 2}$, and let $K = F(\sqrt{a})$. Then $\operatorname{Gal}(K/F) = \langle \sigma \rangle \simeq \mathbb{Z}/2\mathbb{Z}$.

   Let $b = x + y\sqrt{a} \in K^\times \setminus K^{\times 2}$ with $x, y \in F$ and let $L = K(\sqrt{b})$. Show that L is Galois over F if and only if $b\sigma(b) = x^2 - ay^2 \in K^{\times 2}$.

September 2015 Written Certification Exam
Analysis

1. Let $\{f_n\}$ be a sequence of analytic functions converging *pointwise* to a continuous function f on a domain D. Show that f is analytic provided each point $z \in D$ has a neighborhood V such that there is a constant M_V such that $|f_n(z)| \leq M_V$ for all n and $z \in V$.

2. Suppose f has an isolated singularity at z_0.

   (a) Describe the behavior of $|f(z)|$ near z_0 if z_0 is removable or a pole.

   (b) Use the criteria from part (a) to show that if z_0 is an essential singularity for f, then $f\bigl(B'_r(z_0)\bigr)$ is dense in $\mathbf{C}$ for all $r > 0$. Here $B'_r(z_0)$ is the deleted neighborhood $\{\, z \in \mathbf{C} : 0 < |z - z_0| < \epsilon \,\}$. (Hint: suppose to the contrary that there is a $\omega \in \mathbb{C}$ and $r, \epsilon > 0$ such that $|f(z) - \omega| \geq \epsilon$ for all $z \in B'_r(z_0)$. I hope it is clear that you can't evoke the Picard Theorem here.)

3. If X is a topological space, then $\mathcal{B}(X)$ is the σ-algebra generated by the open sets in X — that is, the Borel sets. If $\mathscr{M}$ and $\mathscr{N}$ are sigma algebras in X, then $\mathscr{M} \otimes \mathscr{N}$ is the σ-algebra generated by the measurable rectangles $A \times B$ with $A \in \mathscr{M}$ and $B \in \mathscr{N}$. Show that
$$\mathcal{B}(\mathbf{R} \times \mathbf{R}) = \mathcal{B}(\mathbf{R}) \otimes \mathcal{B}(\mathbf{R})$$
where $\mathbf{R}$ is the real line with its usual topology. (Hint: consider $\mathscr{N} = \{\, A : \mathbf{R} \times A \in \mathcal{B}(\mathbf{R} \times \mathbf{R}) \,\}$.)

4. Let X be a set. Prove that the vector space $B(X, \mathbb{C})$ of bounded complex-valued maps on X is a Banach space for a norm to be specified.

5. Let E be a separable Banach space.

   (a) Recall the definition of the weak topology on E.

   (b) Prove that a norm-convergent sequence is weakly convergent. Does the converse hold?

   (c) Recall the definition of the weak-$*$ topology on E^* and show that a weakly convergent sequence in E^* is also weak-$*$ convergent.

   (d) Prove that a bounded sequence in E^* has a weak-$*$ convergent subsequence.

6. Let S be the map defined on $\ell^2(\mathbb{N})$ by $S(u_0, u_1, \ldots) = (0, u_0, u_1, \ldots)$.

   (a) Prove that S is a bounded linear map between $\ell^2(\mathbb{N})$ and itself and compute $\|S\|$.

   (b) Compute the adjoint S^* of S.

   (c) Is S a normal operator ? Is S an isometry?

September 2015 Written Certification Exam
Topology

1. Show that there is a map $\mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^2$ that is not nullhomotopic. [**Hint:** Consider collapsing the 1-skeleton of a CW structure on the 2-torus to a point.]

2. Let $f : \mathbb{S}^n \to \mathbb{S}^n$ be a map such that $f(\mathbb{D}^n_+) \subseteq \mathbb{D}^n_+$ and $f(\mathbb{D}^n_-) \subseteq \mathbb{D}^n_-$, where $\mathbb{D}^n_\pm$ are the northern and southern hemispheres of $\mathbb{S}^n$. Show that $f(\mathbb{S}^{n-1}) \subseteq \mathbb{S}^{n-1}$, and $\deg(f) = \deg(f|_{\mathbb{S}^{n-1}})$.

3. Let $p : Y \to X$ be a covering map. Let Z be any connected space, and let $f : Z \to X$ be a continuous map. Suppose that $f_1 : Z \to Y$ and $f_2 : Z \to Y$ are continuous lifts of f (i.e., $p \circ f_i = f$ for $i = 1, 2$) that agree at some point $z_0 \in Z$. Show that $f_1 = f_2$ on all of Z.

4. You are given a smooth map of manifolds $\pi : M \to N$ such that every $x \in M$ is in the image of a smooth local section s from a neighborhood $U_{\pi(x)}$ of $\pi(x)$ into a neighborhood of x. Prove that π is a submersion.

5. Show that the 2-sphere S^2 admits a continuous vector field with exactly one zero point.

6. (a) Let $\phi : S^1 \times S^1 \to \mathbb{R}^5$ be a smooth map defined on the 2-torus, where S^1 is the 1-sphere. Let ω be a closed 2-form on $\mathbb{R}^5$. Compute the integral $\int_{S^1 \times S^1} \phi^* \omega$.

   (b) Let $\phi : M \to S^1 \times S^1$ and $\psi : S^1 \times S^1 \to M$ be two smooth maps, where M is a compact oriented 4-manifold. Let ω be a 4-form on M. Compute $\int_M (\psi \circ \phi)^* \omega$.