June 2015 Written Certification Exam
Algebra

1. Let R be a commutative ring with identity (1_R), and let $S \subset R$ be a multiplicatively closed subset of R containing 1_R and such that $0 \notin S$. Let $S^{-1}R$ denote the ring of fractions of R with respect to S.

   (a) For an R-module M, describe the construction of the module of fractions $S^{-1}M$ and its universal mapping property.

   (b) Show that if

   $$L \xrightarrow{\varphi} M \xrightarrow{\psi} N$$

   is an exact sequence of R-modules, then the induced sequence

   $$S^{-1}L \xrightarrow{S^{-1}\varphi} S^{-1}M \xrightarrow{S^{-1}\psi} S^{-1}N$$

   is an exact sequence of $S^{-1}R$-modules.

2. (a) Let R be a PID. Show that a finitely generated R-module is projective if and only if it is free.

   (b) Let R be a commutative ring with identity, and $S = \mathrm{M}_n(R)$ the ring of $n \times n$ matrices with entries in R. Observe that $M = R^n$ (as a column space) is a left S-module. Show that M is a projective S-module, but it is not a free S-module.

3. (a) Compute the order of the group $\mathrm{GL}_n(\mathbb{F}_p)$.

   (b) Show that there is a natural short exact sequence of groups

   $$1 \longrightarrow \mathrm{SL}_n(\mathbb{F}_p) \longrightarrow \mathrm{GL}_n(\mathbb{F}_p) \longrightarrow \mathbb{F}_p^\times \longrightarrow 1\,,$$

   and that the sequence splits.

   (c) Given that the sequence splits, we know that $\mathrm{GL}_n(\mathbb{F}_p)$ is a semidirect product of $\mathrm{SL}_n(\mathbb{F}_p)$ and $\mathbb{F}_p^\times$. Describe this isomorphism.

   (d) Note that part (b) implies that $[\mathrm{GL}_n(\mathbb{F}_p) : \mathrm{SL}_n(\mathbb{F}_p)] = p-1$, so that the size of the Sylow p-subgroups is the same in $\mathrm{SL}_n(\mathbb{F}_p)$ and $\mathrm{GL}_n(\mathbb{F}_p)$. Show that every Sylow p-subgroup of $\mathrm{GL}_n(\mathbb{F}_p)$ is actually a subgroup of $\mathrm{SL}_n(\mathbb{F}_p)$.

4. Let R be a (commutative) domain with 1 containing $\mathbb{C}$ as a subring. Suppose that R is a finite-dimensional $\mathbb{C}$-vector space. Show that $R = \mathbb{C}$.

5. Let K_1, K_2 be finite fields of the same characteristic $\mathrm{char}(K_1) = \mathrm{char}(K_2) = p$, and let $q_1 = \#K_1$ and $q_2 = \#K_2$. Recall that ring homomorphisms are required to map 1 to 1. Show that the following are equivalent:

(i) There is a ring homomorphism $K_1 \to K_2$;

(ii) There is an injective group homomorphism $K_1^\times \hookrightarrow K_2^\times$; and

(iii) q_2 is a power of q_1.

6. The polynomial $f(X) = X^6 - 4X^3 + 1$ is irreducible over $\mathbb{Q}$. Let K be a splitting field for f. Show that

$$\mathrm{Gal}(K/\mathbb{Q}) \cong S_3 \times \mathbb{Z}/2\mathbb{Z}.$$

June 2015 Written Certification Exam
Analysis

1. Suppose that $\{f_n\}$ is a sequence of analytic functions converging uniformly on compact sets of a domain D to a function f. Prove that f is analytic on D.

2. Let f be an entire function such that

$$\lim_{z\to\infty}\left|\frac{f(z)}{z^2}\right| = L < \infty$$

exists. Prove that there are complex constants a, b and c such that $f(z) = az^2 + bz + c$.

3. Recall that Lebesgue measure m on $\mathbf{R}$ has the property that for every measurable set E, we have

$$m(E) = \inf\{\, m(V) : E \subset V \text{ and } V \text{ is open in } \mathbf{R}\,\}. \qquad (\dagger)$$

Given a measurable set E and $\epsilon > 0$, show that there is a closed set F and an open set V such that $F \subset E \subset V$ and $m(V \setminus F) < \epsilon$. [Be careful: it is not immediate from ($\dagger$) that there is an open set V such that $m(V \setminus E) < \epsilon$ unless $m(E) < \infty$.]

4. Let E and F be normed linear spaces over $\mathbb{C}$.

   (a) State the definitions of a bounded linear map between E and F and of the dual space E^*.

   (b) Give an example of a linear map defined everywhere on a normed linear space but that is not bounded.

   (c) Let $T : E \to F$ be a linear map such that for all $\varphi \in F^*$, we have $\varphi \circ T \in E^*$. Prove that T is bounded.

5. Let $\mathcal{A}$ be a unital Banach algebra.

   (a) Consider $a \in \mathcal{A}$ with $\|a\| < 1$. Show that $1_{\mathcal{A}} - a$ is invertible. Does the converse hold?

   (b) Prove that the set of invertible elements in $\mathcal{A}$ is open.

   (c) Let $a \in \mathcal{A}$ and assume that its spectrum $\mathrm{Sp}_{\mathcal{A}}(a)$ is not empty. Prove that $\mathrm{Sp}_{\mathcal{A}}(a)$ is compact.

6. Let $\mathcal{H}$ be a Hilbert space, T in $\mathcal{B}(\mathcal{H})$ and T^* its adjoint.

   (a) Prove that $\ker T = (\operatorname{ran} T^*)^\perp$.

   (b) Let A be a subset of $\mathcal{H}$. Prove that $A^\perp = \overline{\mathrm{span}(A)}^\perp$.

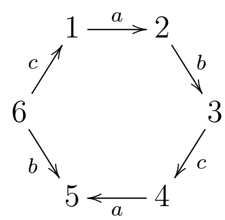
   (c) Show that T is injective if and only if T^* has dense range.

June 2015 Written Certification Exam
Topology

1. Show that any map $\mathbb{S}^2 \to \mathbb{S}^1 \times \mathbb{S}^1$ is nullhomotopic. [You may use without proof that $\mathbb{S}^2$ is simply connected.]

2. Consider the subspace X of $\mathbb{R}^3$ defined by $X = A \cup B$, where
$$A = \{(x, y, 0) : x, y \in \mathbb{R}\}$$
is the xy-plane and
$$B = \{(0, y, z) : y, z \in \mathbb{R}, z \geq 0\}$$
is the upper half of the yz-plane. Show that X is not a topological manifold. [Hint: Consider local homology.]

3. Consider the space X obtained as the quotient space of a planar hexagon and its interior by identifying boundary edges of the hexagon in pairs according to the following scheme, with the indicated orientations of the boundary edges:



(thus, for example, the closed edge joining vertex 2 to vertex 3 is identified with the closed edge joining vertex 6 to vertex 5, etc.). Compute the homology groups of X.

4. Compute the Lie bracket $[V, W]$ of two vector fields
$$V = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \text{ and } W = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}.$$

5. Let M^m be a compact m-dimensional manifold without boundary, with $m \geq 1$. Show that for all $k \geq 1$ there is no submersion $\phi : M \to \mathbb{R}^k$.

6. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth map and $\omega = dx \wedge dy$ be a form on $\mathbb{R}^2$. Let
$$G = \{(x, y, f(x, y)) : (x, y) \in \mathbb{R}^2\} \subset \mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4$$
be the graph of f and let
$$\pi_i : \mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2, \quad i = 1, 2,$$
be the projections to the first and second factors. Let $W = \pi_1^*\omega - \pi_2^*\omega$. Show that $f^*\omega = \omega$ if and only if $W|_{TG} = 0$.