September 2014 Written Certification Exam

Algebra

1. Let A be a 5×5 matrix over $\mathbf{C}$ with minimal polynomial $m_A(x) = x^2(x-2)^2$. What are the possible rational canonical forms and corresponding Jordan forms for A?

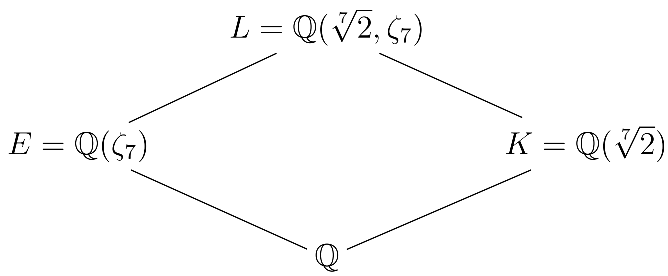
2. Let R be a commutative ring with identity.

   (a) Let M, N be free R-modules. Show that $M \otimes_R N$ is free.

   (b) Let M, N be projective R-modules. Show that $M \otimes_R N$ is projective.

3. Suppose that p and q are distinct primes and that G is a group of order $p^2 q$. Show that G has either a normal p-Sylow subgroup or a normal q-Sylow subgroup.

4. Let $\zeta_7 \in \mathbb{C}$ denote a primitive, complex 7th root of unity. Consider the lattice of fields:



   (a) Compute the degree of each extension in the diagram, justifying your answers.

   (b) Show that L is Galois over $\mathbb{Q}$. Let $G = \mathrm{Gal}(L/\mathbb{Q})$ and H_E, H_K be the subgroups corresponding E and K, respectively. Show explicitly that H_E and H_K are cyclic groups and compute their orders. Using the Galois correspondence, determine the subgroups $H_E \cap H_K$ and $H_E H_K$.

   (c) Show that G is a semidirect product of cyclic groups.

   (d) Let $\sigma \in G$ be characterized by $\sigma(\sqrt[7]{2}) = \sqrt[7]{2}\zeta_7^5$ and $\sigma(\zeta_7) = \zeta_7^3$. What are the fixed fields corresponding to $\sigma H_E \sigma^{-1}$ and $\sigma H_K \sigma^{-1}$?

5. (a) Let $K/\mathbb{Q}$ be a field extension of degree 24. Show that $x^5 + 2x^4 - 16x^3 + 6x - 10$ has no roots in K.

   (b) Show that $\alpha = \sqrt{2} + \sqrt[3]{5}$ is algebraic over $\mathbb{Q}$, and determine the degree of α.

6. (a) Determine the number of distinct roots of the polynomial $x^n - 1$ in an algebraic closure of $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, where p is a prime number and $n > 0$.

   (b) Let $K/\mathbb{Q}$ be a finite extension of fields, and let $\alpha \in K$. Suppose that there is a monic polynomial $f \in \mathbb{Z}[x]$ so that $f(\alpha) = 0$. Show that the minimal polynomial $m_{\alpha,\mathbb{Q}}(x)$ of α over $\mathbb{Q}$ lies in $\mathbb{Z}[x]$.

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Analysis

1. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $\{E_n\}_{n=1}^{\infty}$ be a sequence on $\mathcal{M}$ such that $E_1 \subset E_2 \subset E_3 \ldots$ and let
$$E = \bigcup_{n=1}^{\infty} E_n.$$
Also let $\{A_n\}_{n=1}^{\infty}$ be a sequence on $\mathcal{M}$ such that $A_1 \supset A_2 \supset A_3 \ldots$ and let
$$A = \bigcap_{n=1}^{\infty} A_n.$$
   (a) Suppose that $f : X \to \mathbf{R}$ is integrable. Show that
$$\int_E f\, d\mu = \lim_{n\to\infty} \int_{E_n} f\, d\mu. \tag{1}$$
   and
$$\int_A f\, d\mu = \lim_{n\to\infty} \int_{A_n} f\, d\mu. \tag{2}$$
   (b) Suppose that $f \in L^+(X, \mathcal{M}, \mu)$, i.e., f is a non-negative measurable real-valued function but is not necessarily integrable. Show that Equation (1) is valid for f.
   (c) Give an example of a measure space $(X, \mathcal{M}, \mu)$ and a function $f \in L^+(X, \mathcal{M}, \mu)$ such that Equation (2) fails to hold for f.

2. Suppose that $(X, \mathcal{M}, \mu)$ is a measure space satisfying $\mu(X) < \infty$. Show that $L^q(X, \mathcal{M}, \mu) \subset L^p(X, \mathcal{M}, \mu)$ whenever $0 < p < q$. (Be sure to include the case that $q = \infty$.)

3. Let Γ_R^+ be the semicircle defined by $|z| = R$ and $\mathrm{Im}(z) \geq 0$, and let Γ_R^- be the semicircle given by $|z| = R$ and $Im(z) \leq 0$. Give both semicircles the counterclockwise orientation.
   (a) Evaluate
$$\lim_{R\to\infty} \int_{\Gamma_R^+} \frac{e^{iz}}{z^4} dz.$$
   (b) Evaluate
$$\lim_{R\to\infty} \int_{\Gamma_R^-} \frac{e^{iz}}{z^4} dz.$$

4. Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed vector spaces. Give the Cartesian product
$$V \times W = \{(x, y) \,|\, x \in V, y \in W\}$$

the obvious coordinate-wise defined vector space structure and the norm

$$\|(x,y)\|_{V\times W} = \|x\|_V + \|y\|_W \quad \text{for } x \in V, y \in W.$$

Prove that the graph

$$G(T) = \{(x, T(x)) : x \in V\}$$

of a continuous, linear mapping $T : V \to W$ is a closed, linear subspace of $(V \times W, \|\cdot\|_{V\times W})$.

5. Let c denote the $\mathbb{C}$-vector space of all convergent complex sequences. Show that c is a Banach space when equipped with the supremum norm from ℓ^∞:

$$\|(x_n)\|_\infty = \sup_{n\in\mathbb{N}} |x_n|.$$

6. Let $\mathcal{H}$ be a separable infinite-dmensional Hilbert space. Consider the set

$$F(\mathcal{H}) = \{T \in B(\mathcal{H}) : \dim(\text{range}(T)) < \infty\}$$

of bounded *finite rank* operators. It is easy to see that $F(\mathcal{H})$ is a subalgebra of the algebra $B(\mathcal{H})$, but it has further structure.

a.) Show that if $T \in F(\mathcal{H})$ then $T^* \in F(\mathcal{H})$ and $\dim(\text{range}(T^*)) = \dim(\text{range}(T))$.

b.) Show that if $T \in F(\mathcal{H})$ then $ST, TS \in F(\mathcal{H})$ for all $S \in B(\mathcal{H})$.

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Topology

1. Let $C : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ by $C(v, w) = v \times w$, the usual vector cross product. Determine the critical points of C. Conclude that for $0 \neq u \in \mathbb{R}^3$, the set $\{(v, w) \in \mathbb{R}^3 \times \mathbb{R}^3 : v \times w = u\}$ is a smooth manifold. If $\{e_1, e_2, e_3\}$ denotes the standard basis for $\mathbb{R}^3$, determine a basis of the tangent space $T_{(e_1,e_2)}(C^{-1}(e_3))$ as a vector subspace of $T_{(e_1,e_2)}(\mathbb{R}^3 \times \mathbb{R}^3) \cong \mathbb{R}^6$.

2. Let $p : Y \to X$ be a covering map. Let Z be any connected space, and let $f : Z \to X$ be a continuous map. Suppose that $f_1 : Z \to Y$ and $f_2 : Z \to Y$ are continuous lifts of f (i.e., $p \circ f_i = f$ for $i = 1, 2$) that agree at some point $z_0 \in Z$. Show that $f_1 = f_2$ on all of Z.

3. Consider the circle $\mathbb{S}^1$ with its usual CW-structure with a single 0-cell e^0 and a single 1-cell e^1. Let X be the space obtained from $\mathbb{S}^1$ by attaching two 2-cells e_1^2 and e_2^2 by maps of degree 2 and degree 3, respectively. Compute the homology groups of X.

4. Let $f \in \mathbb{R}[X, Y, Z]$ be a homogeneous quadratic polynomial with real coefficients. Let $\mathbb{D}^3 = \{x \in \mathbb{R}^3 : \|x\| \leq 1\}$ be the unit disk and $\mathbb{S}^2 = \{x \in \mathbb{R}^3 : \|x\| = 1\}$ the unit sphere in Euclidean space $\mathbb{R}^3$. Let ν denote the volume form on $\mathbb{S}^2$, where $\mathbb{S}^2$ is given the orientation induced from the standard orientation of $\mathbb{D}^3$. Prove that

$$\int_{\mathbb{D}^3} \Delta f = 2 \int_{\mathbb{S}^2} f\, \nu,$$

where Δf is the Laplacian $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$. [**Hint:** Write Δf as a divergence.]

5. Let $\mathbb{RP}^n$ denote real projective n-space, the quotient space of $\mathbb{R}^{n+1} - \{0\}$ obtained from the equivalence relation identifying two nonzero vectors if they span the same line. Show that if $n > 0$ is even, then every continuous map $f : \mathbb{RP}^n \to \mathbb{RP}^n$ has a fixed point. [**Hint**: Translate the problem to one about mappings $\mathbb{S}^n \to \mathbb{S}^n$; recall the proof that an even-dimensional sphere has no nowhere-vanishing vector field. You may use without proof that $\mathbb{S}^n$ is simply connected for $n > 1$.]

6. Let M be a smooth n-manifold whose smooth structure is defined by a maximal atlas $\mathscr{M}$ of charts (x, U), where $U \subseteq M$ is open and $x : U \to x(U) \subseteq \mathbb{R}^n$ is a homeomorphism of U with an open subset of $\mathbb{R}^n$. Suppose that there is a nowhere-vanishing smooth n-form $\omega \in \Omega^n(M)$. Show that there is a subcollection $\mathscr{A}$ of $\mathscr{M}$ such that

   - The collection $\{U : (x, U) \in \mathscr{A}\}$ is an open cover of M.
   - For any two overlapping charts charts (x, U), $(y, V) \in \mathscr{A}$, i.e., any two charts in $\mathscr{A}$ such that $U \cap V \neq \varnothing$, the derivative $D(y \circ x^{-1})(x(p)) : \mathbb{R}^n \to \mathbb{R}^n$ of the mapping $y \circ x^{-1} : x(U \cap V) \to y(U \cap V)$ has positive determinant for every $p \in U \cap V$.