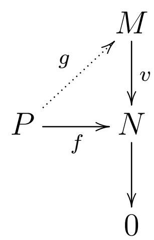
June 2014 Written Certification Exam

Algebra

1. Let R be a commutative ring. An R-module P is *projective* if for all R-module homomorphisms $v : M \to N$ and $f : P \to N$ with v surjective, there exists an R-module homomorphism g lifting f in the sense that the diagram



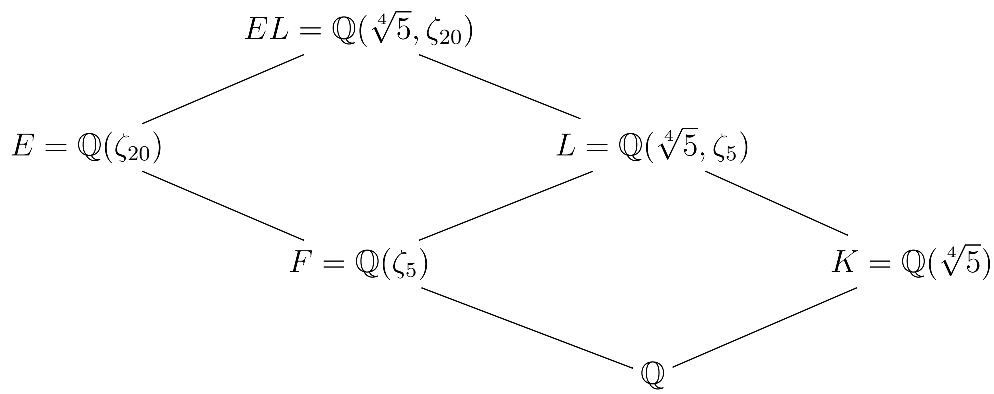
commutes. Show that an R-module P is projective if and only if P is a direct summand of a free R-module.

2. Let $A \in M_n(\mathbf{C})$ (with $n \geq 1$) and suppose $A^m = 0$ for some $m > 0$. Show that $A^n = 0$ and that the trace of A is zero.

3. Suppose that G is a group of order 105 and that G has a normal 3-Sylow subgroup. Show that G is cyclic.

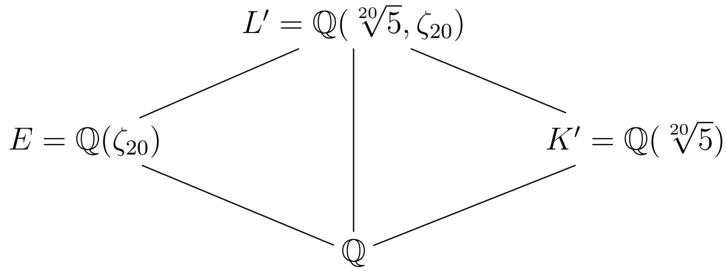
4. For $n \geq 1$, let $\zeta_n \in \mathbb{C}$ denote a primitive nth root of unity.

   (a) Consider the following lattice of fields.



   Determine the degrees of all the extensions, giving reasons for your answers.

(b) Now consider the lattice



For each of the five extensions in this diagram, determine whether the extension is Galois or not. If the extension is Galois, determine the strongest adjective that describes the Galois group from among the following list (in ascending order):

$$\text{nonabelian} \Leftarrow \text{solvable} \Leftarrow \text{abelian} \Leftarrow \text{cyclic}$$

and offer a brief explanation. (You may compute Galois groups to justify your assertions, but this is not required.)

5. (a) Suppose K_1/F and K_2/F are finite Galois field extensions. Show that the extensions $(K_1 \cap K_2)/F$ and K_1K_2/F are Galois.

   (b) Prove that for any integer $n \geq 1$ there is a Galois extension $K/\mathbb{Q}$ with $[K : \mathbb{Q}] = n$.

6. Let A be a Noetherian integral domain (which is not a field), B a commutative ring with identity.

   (a) Let $a \in A$ be a nonzero, non-unit. Show that a can be written as a finite product of irreducibles.

   (b) Let $\varphi : A \to B$ be a surjective ring homomorphism. Show that B is a Noetherian ring.

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Analysis

1. Let $(X, \mathcal{M})$ be a measurable space.

   (a) Let $\{f_n\}$ be a sequence of measurable functions with $f_n : X \to [-\infty, \infty]$. Show that the function
   $$g : X \to [-\infty, \infty]$$
   $$g(x) = \sup\{f_n(x) : n \geq 1\}$$
   is measurable.

   (b) For $\{f_n\}$ as in part (a), show that the function
   $$h : X \to [-\infty, \infty]$$
   $$h(x) = \limsup\{f_n(x) : n \geq 1\}$$
   is measurable.

   (c) Let $f : X \to \mathbf{R}$ and $g : X \to \mathbf{R}$ be measurable functions. Let
   $$E = \{x \in X : f(x) > g(x)\}.$$
   Starting with the definition of measurable function, show that $E \in \mathcal{M}$.

2. Let $(X, \mathcal{M}, \mu)$ be a measure space, let $\mathcal{N}$ be a σ-algebra on a set Y, and let $f : X \to Y$ be an $(\mathcal{M}, \mathcal{N})$-measurable function. Define $\nu : \mathcal{N} \to [0, \infty]$ by
   $$\nu(A) = \mu(f^{-1}(A))$$
   for all $A \in \mathcal{N}$.

   (a) Show that ν is a measure on $(Y, \mathcal{N})$.

   (b) For $g \in L^+(Y, \mathcal{N})$ (i.e., $g : Y \to [0, \infty]$ is a measurable function), show that
   $$\int_Y g\, d\nu = \int_X g \circ f\, d\mu.$$
   (Suggestion: First verify the statement when g is the characteristic function of a measurable set.)

3. Let $f : \mathbf{C} \to \mathbf{C}$ be an entire function.

   (a) Prove that if f is nonconstant, then the image of f is dense in $\mathbf{C}$.

   (b) Suppose
   $$\lim_{z \to \infty} f(z) = \infty.$$
   Show that f is a polynomial.

4. Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed vector spaces. Equip the vector space direct product

$$V \times W = \{(x, y) : x \in V \text{ and } y \in W\}$$

with the norm

$$\|(x, y)\|_1 = \|x\|_V + \|y\|_W \quad \text{for } x \in V \text{ and } y \in W.$$

Suppose that $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ are Banach spaces. Prove that $(V \oplus W, \|\cdot\|_1)$ is also a Banach space.

5. Consider the Hilbert space $L^2(0, 1)$ with respect to Lebesgue measure on the open unit interval $(0, 1)$. For each $f \in L^2(0, 1)$ define

$$M(f) : (0, 1) \to \mathbf{R}$$
$$M(f)(x) = xf(x)$$

   (a) Show that M is a well-defined bounded linear operator on $L^2(0, 1)$.

   (b) Show that M is injective and M is *self-adjoint*, i.e.

$$(M(f), g) = (f, M(g))$$

   for all $f, g \in L^2(0, 1)$.

6. Let $(V, \|\cdot\|)$ be a normed vector space over a field $\mathbb{F}$. Let $M \subsetneq V$ be a proper closed subspace of V and let $x \in V \backslash M$.

   (a) Show that $\delta = \inf\{\|x - y\| : y \in M\} > 0$.

   (b) Show that there exists a bounded linear functional $f \in V^*$ such that $\|f\| = 1$ and $f(x) = \delta$ and $f|_M = 0$. *(Hint: Work with $M + \mathbb{F}x$.)*

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Topology

1. Let M be a smooth manifold, let $x_0, x_1 \in M$, and let $\alpha, \beta : [0,1] \to M$ be smooth paths such that $x_0 = \alpha(0) = \beta(0)$ and $x_1 = \alpha(1) = \beta(1)$. We say α is *smoothly path homotopic* to β if there exists a smooth map $h : [0,1] \times [0,1] \to M$ satisfying the conditions

   - For all $s \in [0,1]$, we have $h(s,0) = \alpha(s)$ and $h(s,1) = \beta(s)$.
   - For all $t \in [0,1]$, we have $h(0,t) = x_0$ and $h(1,t) = x_1$.

   Let $\omega \in \Omega^1(M)$ be a closed smooth 1-form on M. Show that if α is smoothly path homotopic to β, then
   $$\int_\alpha \omega = \int_\beta \omega.$$

2. Consider S be the surface
   $$S = \{(x,y,z) \in \mathbb{R}^3 : x - yz + z^3 = 0\} \subseteq \mathbb{R}^3$$
   Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be the projection $(x,y,z) \mapsto (x,y)$. Let H be the collection of points $p \in S$ such that $\pi|_S : S \to \mathbb{R}^2$ is *not* a local diffeomorphism in a neighborhood of p. Show that H is a smooth curve in $\mathbb{R}^3$ and determine a parametrization for it.

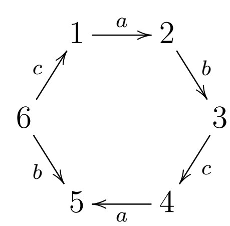
3. Let M be a smooth n-manifold with smooth atlas of charts $\mathscr{A}$. Suppose that for all charts (x,U) and (y,V) in $\mathscr{A}$ with $U \cap V \neq \emptyset$, the change of charts map
   $$y \circ x^{-1} : x(U \cap V) \to y(U \cap V)$$
   has derivative
   $$D(y \circ x^{-1})(x(p)) : \mathbb{R}^n \to \mathbb{R}^n$$
   with positive determinant for every $p \in U \cap V$. Show that there is a nowhere vanishing smooth n-form $\omega \in \Omega^n(M)$.

4. Let X and Y be topological spaces. Let $X \sqcup Y$ denote the disjoint union of X and Y, endowed with the disjoint union or coproduct topology, and let $i_X, i_Y : X, Y \to X \sqcup Y$ be the natural inclusion maps.

   For any homology theory H satisfying the Eilenberg-Steenrod axioms, prove that induced maps
   $$i_{X*} : H_q(X) \to H_q(X \sqcup Y) \text{ and } i_{Y*} : H_q(Y) \to H_q(X \sqcup Y)$$
   induce an isomorphism
   $$H_q(X) \oplus H_q(Y) \xrightarrow{\sim} H_q(X \sqcup Y)$$
   for each $q \geq 0$.

5. Let $f : \mathbb{S}^2 \to \mathbb{T}^2$ be any continuous map from the 2-sphere $\mathbb{S}^2$ to the 2-torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$. Show that f is homotopic to a constant map.

6. Consider the topological space X obtained as the quotient space of a planar hexagon and its interior by identifying boundary edges of the hexagon in pairs according to the following scheme, with the indicated orientations of the boundary edges:



(For example, the closed edge joining vertex 2 to vertex 3 is identified with the closed edge joining vertex 6 to vertex 5). Compute the homology groups $H_q(X)$ for $q \geq 0$.